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BILATERAL MATCHING WITH LATIN SQUARES

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Bilateral Matching with Latin Squares[*]

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ABSTRACT: We develop a general procedure to construct pairwise meeting processes character-
ized by two features. First, in each period the process maximizes the number of matches in the
population. Second, over time agents meet everybody else exactly once. We call this type of
meetings "absolute strangers." Our methodological contribution to economics is to offer a simple
procedure to construct a type of decentralized trading environments usually employed in both
theoretical and experimental economics. In particular, we demonstrate how to make use of the
mathematics of *Latin Squares* to enrich the modeling of matching economies.

Keywords and Phrases: Latin squares, Matching models, Spatial interactions

JEL Classification Numbers: C00, C78, D83, E00

1. Introduction

This paper offers a simple procedure that can be used to construct sequences of pairwise
meetings among players drawn from a finite population. The meeting process that we are
interested in studying has two properties. First, the sequence of meetings is exogenous
and it is such that players meet everybody else exactly once. Second, in each period the
process maximizes the number of matches in the population.

Pairwise meeting processes of this type are often used in economics to make explicit a
notion of trade frictions. For example, they are used in macroeconomics to model obstacles
to the exchange process, as in the random matching model in [6], in monetary economics to
introduce obstacles to credit transactions, as in the deterministic pairwise matching model
in [10] or the random matching model in [7], and in experimental economics to introduce

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informational isolation, as in [4]. Indeed, we will borrow terminology from experimental economics (see [2] and [8]) and refer to agents matched in the manner described above as being 'absolute strangers.'

To develop a procedure to create the desired sequence of matches, we use a special class of permutations called involutions, and we exploit some results from the mathematics of Latin squares. In a nutshell, the reason for working with these mathematical objects is the following. A matching process is a way to repeatedly partition a population X into disjoint sets of agents (for a formalization see [1]). Therefore, since the meetings we consider are bilateral, a matching process can be viewed as a sequence of involutions from X to X. Indeed, an involution is simply a permutation such that the function composed with itself is the identity function. It turns out that constructing the desired type of bilateral matching process can be conveniently done by arraying involutions by means of Latin squares. A *Latin square* is an $n \times n$ matrix filled with n different symbols arranged in such a way that each symbol appears exactly once in every row and column.[1] We will interpret symbols as agents and then we will offer a procedure to match 'absolute strangers' by demonstrating how to construct Latin squares such that all rows, but the initial, are involutions of the first row.

To do so we take several steps. First, we explain how to create absolute strangers meetings among agents who belong to two different but equally sized groups. Subsequently, we study how to create matches of this type when agents belong to an odd-sized group, ensuring that everyone remains unmatched exactly once. Finally, we exploit the two earlier steps to demonstrate how to obtain absolute strangers pairings on any finite population. In particular, we prove that, given a population of size n, we can create exactly $n-1$ matching rounds among absolute strangers. Besides offering a new method of constructing and formalizing pairwise matching economies, our procedure has practical applications in the design of experimental matching economies. Indeed, our construction scheme is simple, because it can be accomplished quickly with pencil and paper, and so it allows to devise a desired pairing scheme without having to use specialized software.

The paper is organized as follows. Section 2 introduces the mathematical background. Section 3 discusses the interpretation of Latin squares as absolute strangers bilateral matching processes. Section 4 and Section 5 show the existence and the construction of pairwise matches of the type desired. Section 6 provides a practical example that might of interest in experimental economics. Section 7 concludes with some final remarks.

2. MATHEMATICAL BACKGROUND: LATIN SQUARES

We discuss here the basic mathematical concepts that are needed to formalize our notion of pairwise matching. The most important one is that of a Latin square.

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[1]When Leonhard Euler started to study Latin squares in 1782, he used Latin characters as symbols; and hence the origin of the name. Latin squares have been used especially to design agricultural experiments and to construct tournaments; see [9].

Definition 2.1. *Given n distinct symbols, a **Latin square** is an $n \times n$ matrix with entries from the given symbols arranged in such a way that every symbol appears exactly once in each row and in each column.*

Given the sets of symbols $\{1, 2, 3, 4\}$ and $\{\bigstar, \maltese, \spadesuit, \clubsuit\}$, the matrices

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\bigstar & \maltese & \spadesuit & \clubsuit \\
\spadesuit & \clubsuit & \bigstar & \maltese \\
\maltese & \spadesuit & \clubsuit & \bigstar \\
\clubsuit & \bigstar & \maltese & \spadesuit
\end{bmatrix},
$$

are two examples of Latin squares. Of course, for a given set of n symbols we generally have many different Latin squares. Indeed, the number of distinct Latin squares grows so rapidly with n that, although Latin squares have been studied extensively in mathematics (see, for instance [5, 9]), the number of distinct Latin squares has been calculated only for up to $n = 10$.

Our purpose here is to identify three basic types of Latin squares that will be the building blocks for our matching processes. Given a population set $X = \{1, \ldots, n\}$ with n agents, in what follows we introduce three Latin square constructions each of which generates a specific $n \times n$ matrix.

Latin Square Construction #1

This Latin square is denoted L^- and its first row is the vector $(1, 2, \ldots, n)$. The other rows of L^- are generated recursively by shifting by one position to the right the previous row in a cyclical manner. Thus, the second row is obtained by shifting by one position to the right the first row, i.e., the second row is the vector $(2, 3, \ldots, n, 1)$ and the third row is the vector $(3, 4, \ldots, n, 1, 2)$, etc. Specifically, L^- is the following $n \times n$ Latin square:

$$
L^- = \begin{bmatrix}
1 & 2 & \cdots & n-2 & n-1 & n \\
2 & 3 & \cdots & n-1 & n & 1 \\
3 & 4 & \cdots & n & 1 & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n-1 & n & \cdots & n-4 & n-3 & n-2 \\
n & 1 & \cdots & n-3 & n-2 & n-1
\end{bmatrix}
$$

If we use the standard notation $L^- = [a_{ij}]$ to denote this Latin square, then it is easy to see that its entries a_{ij} are given by the formula

$$
\begin{aligned}
a_{ij} &= i + j - 1 - n\chi_Y(i + j - 1) \\
&= \begin{cases}
j & \text{if } i = 1 \text{ and } 1 \leq j \leq n \\
i + j - 1 & \text{if } i \geq 2 \text{ and } 1 \leq j \leq n - i + 1 \\
j - (n - i) - 1 & \text{if } i \geq 2 \text{ and } n - i + 2 \leq j \leq n,
\end{cases}
\end{aligned}
$$

where $\chi_Y : \mathbb{N} \to \{0,1\}$ is the charecteristic function of the set

$$Y = \{n+1, n+2, \ldots\},$$

defined as usual by $\chi_Y(k) = 1$ if $k \in Y$ and $\chi_Y(k) = 0$ if $k \notin Y$.

For instance, when $n = 4$ this construction yields the Latin square

$$L^- = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}.$$

Latin Square Construction $\#\,2$

We denote this Latin square by $L^+ = [a_{ij}]$. It has first row the vector $(1, 2, \ldots, n)$ and its construction is done recursively exactly as in the Latin square of Construction $\#\,1$ with the only difference that this time we shift to the left. This means that the second row of L^+ is obtained by shifting by one position to the left the first row in a cyclical manner, i.e., the second row is $(n, 1, \ldots, n-2, n-1)$, and the third row is the vector $(n-1, n, 1, \ldots, n-2)$, etc. The complete Latin square L^+ is the following:

$$L^+ = \begin{bmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ n & 1 & \cdots & n-3 & n-2 & n-1 \\ n-1 & n & \cdots & n-4 & n-3 & n-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 3 & 4 & \cdots & n & 1 & 2 \\ 2 & 3 & \cdots & n-1 & n & 1 \end{bmatrix}$$

An easy verification shows that the entries a_{ij} of L^+ are given by the formula

$$\begin{aligned} a_{ij} &= n+1+j-i-n\chi_Y(n+1+j-i) \\ &= \begin{cases} j & \text{if } i=1 \text{ and } 1 \le j \le n \\ (n-i)+j+1 & \text{if } i \ge 2 \text{ and } 1 \le j \le i-1 \\ j-i+1 & \text{if } i \ge 2 \text{ and } i \le j \le n. \end{cases} \end{aligned}$$

When $n = 4$ we have:

$$L^+ = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Latin Square Construction #3

This is the Latin square $L = [a_{ij}]$ when the first row is the vector $(n, n-1, \ldots, 1)$ and the other rows of L are constructed by following the recursive procedure of the Construction #1. That is, the second row is obtained by shifting by one position to the right the first row, i.e., the second row is $(n-1, n-2, \ldots, 1, n)$. Repeating this shifting process $n-1$ times recursively we obtain the following $n \times n$ Latin square:

$$
L = \begin{bmatrix}
n & n-1 & \cdots & 2 & 1 \\
n-1 & n-2 & \cdots & 1 & n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
n-i+1 & n-i & \cdots & n-i+3 & n-i+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 1 & \cdots & 4 & 3 \\
1 & n & \cdots & 3 & 2
\end{bmatrix}
$$

As before, an easy verification shows that the entries a_{ij} of L^+ are given by the formula

$$
\begin{aligned}
a_{ij} &= n + 1 - (i + j - 1) + n\chi_Y(i + j - 1) \\
&= \begin{cases}
n + 1 - j & \text{if } i = 1 \text{ and } 1 \le j \le n \\
n + 2 - i - j & \text{if } i \ge 2 \text{ and } 1 \le j \le n - i + 1 \\
2n - i + 2 - j & \text{if } i \ge 2 \text{ and } n - i + 2 \le j \le n.
\end{cases}
\end{aligned}
$$

When $n = 4$ the Latin square L is the following:

$$
L = \begin{bmatrix}
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 4 \\
2 & 1 & 4 & 3 \\
1 & 4 & 3 & 2
\end{bmatrix}
$$

3. Matching Matrices and Latin Squares

The three types of Latin squares that we have presented above are useful to model the desired type of bilateral matchings among agents in a finite population. A **bilateral matching** on a population X is simply a function $\phi \colon X \to X$ satisfying $\phi^2(x) = x$ for all $x \in X$, that is ϕ is a special type of permutation called an involution.[2] We interpret agent $\phi(x)$ to be the partner of agent x and so a sequence of meetings (or bilateral matchings) is simply a sequence of bilateral matchings. Of course, every bilateral

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[2]For further details see [1].

matching ϕ automatically provides a partition of the population X into pairs—this is the partition $(\{x, \phi(x)\})_{x \in X}$.

In this section we are interested in a particular type of matching process. Specifically, we wish to match everyone in the population to someone else, whenever this is feasible, but we also want to ensure that agents meet everybody else in the population exactly once. These requirements immediately imply that, if we have a population of n agents, the desired matching process cannot last more than $n - 1$ periods, since each agent can be matched at most to $n - 1$ different individuals. To formalize such a matching process, we need to introduce a special type of matrix.

Definition 3.1. *Let* $X = \{1, \ldots, n\}$ *be a population. An* $m \times n$ *matrix* $\mathrm{M} = [\pi_{ij}]$ *with entries from the population is called a **matching matrix** if:*
  (1) *Its first row is the vector* $(1, 2, \ldots, n)$.
  (2) *Each other row is an involution of the first row.*
  (3) *If* n *is even every column has distinct entries.*
  (4) *If* n *is odd, then in each column* j *the agent* j *appears at most twice and the remaining entries in the column are all distinct (and hence they are precisely the agents* $X \setminus \{j\}$*).*

An $m \times n$ matching matrix M is called **maximal** if:
  (a) When n is even M has n rows, i.e., $m = n$.
  (b) When n is odd M has $n + 1$ rows, i.e., $m = n + 1$.[3]

Given a population $X = \{1, \ldots, n\}$ we will see that several distinct maximal matching matrices can be constructed. For a population X of size n we denote an arbitrary maximal matching matrix by $\mathcal{M}_n(X)$, omitting the argument X when it is understood. To emphasize the link between maximal matching matrices and meeting processes, we introduce the following terminology.

Definition 3.2. *An **absolute strangers** matching process for a population is a maximal matching matrix.*

To see why a maximal matching matrix represents a sequence of partitions of the population into the type of pairwise matches that we desire, consider populations with $n = 3$ and $n = 4$. Two corresponding maximal matching matrices are:

$$
\mathcal{M}_3 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathcal{M}_4 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}
$$

The first row simply lists all agents of the population X, which we order from $1, 2, \ldots, n$. Clearly, each subsequent row defines a partition of the population into pairs, in some

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[3] In each column j the agent j appears exactly twice and the remaining $m - 2$ entries are all distinct.

period. To see how this is done, let $t = 1, \ldots, n - 1$ denote a matching period, and $t = 0$ denote an initial stage where no one is matched. So, each row i pinpoints a distinct matching period $t = i - 1$. The partition in period t is thus identified by associating to each element in column j of the first row, the element present in the same column of row $t + 1$. That is, we read the matches vertically.

For instance, the matching matrix $\mathcal{M}_4$ above describes the following sequence of three pairwise meetings on the population $X = \{1, 2, 3, 4\}$:

$$
\begin{array}{c}
t \\ 1 \\ 2 \\ 3
\end{array}
\left[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{array}
\right]
$$

Consider the second row, i.e., period $t = 1$. Agent 1 is matched to agent 2, and column two confirms that agent 2 is matched to agent 1. The other two columns tell us that agent 3 is matched to agent 4 in period $t = 1$. Subsequent periods are interpreted similarly.

The (maximal) matching matrix $\mathcal{M}_n$ conveniently describes pairwise encounters among n agents such that no one meets the same partner again, which is a feature of several matching frameworks in economics.[4]

It is important to recognize that for n even any matrix $\mathcal{M}_n$ is a Latin square that satisfies the additional restriction that every row is an involution of the first. This is a special case, since not all Latin squares possess this property. For instance, in the Latin square below

$$
\left[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{array}
\right],
$$

while each number appears exactly once in each column and each row, the Latin square does not represent a matching for the population $X = \{1, 2, 3, 4\}$. Notice that the second row is not an involution of the first since in period $t = 1$ agent 1 appears to be matched to agent 2 (first column) but agent 2 is matched to agent 3 (second column).

Clearly, for n odd a maximal matching matrix is not a Latin square because $m = n + 1$, so it is not a square matrix. However, notice that every element j appears in the first row of column j, and exactly once more below, in that same column. This implies that if n is odd, then by eliminating the first row from the matrix $\mathcal{M}_n$ we obtain a Latin square, something that will come in handy to construct matchings on any population.

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[4]For instance, it reflects the meeting restrictions assumed in the pairwise random matching model of Kiyotaki and Wright [7], the spatial separation requisite described in the multilateral matching model in [3], the 'anonymity' requisites studied in [1], and the informational isolation in the experimental study in [4]. Indeed, the name we use to describe the meeting process reflects a description of a matching protocol found in a popular experimental software platform [8], which further restricts the interactions possible under the 'strangers' matching protocol (see [2]).

For example, when $n = 3$, we can see that the matrix to the right below, obtained by eliminating the first row from $\mathcal{M}_3$ is a Latin square.

$$\mathcal{M}_3 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

4. Maximal Matching Matrices

In this section we show existence of maximal matching matrices for any finite population. We start by pairing agents across two groups of equal size. Suppose we have a population composed of two groups, denoted A and B, of n individuals each. For practical purposes, we interpret each group as being composed by a homogenous type of agents, such as buyers or sellers. Our objective is to pair exactly once each agent from A to someone from B, so that everyone in a group is always matched to someone else in the other group. That is, we want to pair agents across groups in an absolute strangers fashion.

It is immediate that this matching protocol can generate at most n periods of matching since each agent can meet at most n agents from the other group. The key questions are whether we can sustain n matching periods and, if we can, if there is a systematic way[5] to match agents as absolute strangers.

The following result achieves two goals. On one hand, it establishes that we can pair agents as absolute strangers at most n times. And on the other hand, it offers a basic procedure for constructing the desired matching protocol on any population. Note that the notation $X = A \sqcup B$, means $X = A \cup B$ and $A \cap B = \emptyset$, i.e., X is the disjoint union of A and B.

Lemma 4.1. *Let $A = \{1, \ldots, n\}$ and $B = \{n+1, \ldots, 2n\}$. Then the $(n+1) \times n$ matrix*

$$\mathrm{M}(A, B) = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \cdots & \mathbf{n-1} & \mathbf{n+2} & \mathbf{n+1} & \mathbf{n+2} & \cdots & \mathbf{2n-1} & \mathbf{2n} \\ n+1 & n+2 & \cdots & 2n-1 & 2n & 1 & 2 & \cdots & n-1 & n \\ n+2 & n+3 & \cdots & 2n & n+1 & n & 1 & \cdots & n-2 & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n & n+1 & \cdots & 2n-2 & 2n-1 & 2 & 3 & \cdots & n & 1 \end{bmatrix}$$

is a matching matrix for the population $X = A \sqcup B$ such that every agent in A is pairwise matched to every agent in B.

The proof of this lemma is straightforward. Start by noticing that the notation $\mathrm{M}(A, B)$ indicates that we are matching agents from set A to set B, without matching them within sets. Therefore, it is easy to see that the matrix $\mathrm{M}(A, B)$ is a desired matching matrix

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[5]By a "systematic way" we mean, as usual, the development of an algorithm that can be executed by a computer.

for the population $X = A \sqcup B$. Also, let $L^-(B)$ be the $n \times n$ Latin square with symbols from $B = \{n+1, \ldots, 2n\}$ (constructed using procedure #1 in Section 2). Similarly, $L^+(A)$ denotes the $n \times n$ Latin square with symbols from $A = \{n+1, \ldots, 2n\}$ (constructed using procedure #2 in Section 2). Then we can rewrite $M(A, B)$ as follows

$$
M(A, B) = \left[\begin{array}{ccccccc} \mathbf{1} & \cdots & \mathbf{n} & \mathbf{n+1} & \cdots & \mathbf{2n} \\ & L^-(B) & & & L^+(A) & \end{array} \right].
$$

The reader should observe that a matching matrix obtained by the construction provided in Lemma 4.1 is not maximal, since it provides matches across the groups A and B, but not within each group. An example follows.

Let $X = \{1, \ldots, 8\}$ with $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$. Then, it is easy to see that the procedure above gives the following 5×8 matching matrix

$$
M(A, B) = \left[\begin{array}{cccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 5 & 4 & 1 & 2 & 3 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 8 & 5 & 6 & 7 & 2 & 3 & 4 & 1 \end{array} \right].
$$

We are now ready to state and prove the main result of this paper.

Theorem 4.2. *Every finite population admits a maximal matching matrix.*

Proof. The proof consists of two parts. In the first part we construct a maximal matching matrix for odd populations, while the second part shows existence of maximal matching matrices for any even population.

Let us start with a population $X = \{1, \ldots, n\}$, where n is odd. Recall that L is the Latin Square constructed using procedure #3 in Section 2. Now, notice that the $(n+1) \times n$ matrix

$$
\begin{aligned}
\mathcal{M}_n &= \left[\begin{array}{ccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & \mathbf{n-2} & \mathbf{n-1} & \mathbf{n} \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \\ n-1 & n-2 & n-3 & \cdots & 2 & 1 & n \\ n-2 & n-3 & n-4 & \cdots & 1 & n & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 1 & n & \cdots & 5 & 4 & 3 \\ 1 & n & n-1 & \cdots & n-1 & 3 & 2 \end{array} \right] \\
&= \left[\begin{array}{ccc} \mathbf{1} & \cdots & \mathbf{n} \\ & L & \end{array} \right]
\end{aligned}
$$

is a maximal matching matrix for the population X.

In the second part let $X = \{1, \ldots, 2n\}$. Write $2n = 2^k p$, where p, k are natural numbers with p odd. We need to consider two cases.

Case 1: $p = 1$.

In this case we have $2n = 2^k$. The existence of a maximal matching matrix will be established by induction on k. For $k = 1$, the 2×2 matrix $\mathcal{M}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is a maximal matching matrix.

Now assume that a maximal matching matrix exists for a population $2n = 2^k$, where $k \geq 1$. We need to show that a maximal matching matrix exists for $2n = 2^{k+1}$. To this end, let $X = \{1, \ldots, 2^{k+1}\} = A \sqcup B$, where $A = \{1, \ldots, 2^k\}$ and $B = \{2^k + 1, \ldots, 2^{k+1}\}$.

For the induction step, we know that there exists a $2^k \times 2^k$ maximal matching matrix for the population A, say $\mathcal{M}_{2^k}(A)$. Similarly, there exists a $2^k \times 2^k$ maximal matching matrix for the population B, say $\mathcal{M}_{2^k}(B)$.

It is not difficult to check that the $2^{k+1} \times 2^{k+1}$ matrix

$$\mathcal{M}_{2^{k+1}} = \begin{bmatrix} \mathcal{M}_{2^k}(A) & \mathcal{M}_{2^k}(B) \\ \mathcal{M}_{2^k}(B) & \mathcal{M}_{2^k}(A) \end{bmatrix}$$

is a maximal matching matrix for the population X. (For another construction of a maximal matching matrix for X see also Lemma 5.1 below.)

Case 2: $p > 1$.

In this case we have $2n = 2^k p$, and we can still use induction on k. Let p be fixed. For $k = 1$, $2n = 2p$ and so $X = \{1, \ldots, 2p\}$. Let $X = A \sqcup B$ where $A = \{1, \ldots, p\}$ and $B = \{p+1, \ldots, 2p\}$, i.e., A and B have p agents each. Let $L^-(B)_{-1}$ and $L^+(A)_{-1}$ denote the $(p-1) \times p$ Latin squares obtained by deleting the first row of $L^-(B)$ and $L^+(A)$.

Next, notice that p is odd and so we can use the first part to construct the maximal matching matrices $\mathcal{M}_p(A)$ and $\mathcal{M}_p(B)$. Let $\mathcal{M}'_p(A)$ and $\mathcal{M}'_p(B)$ denote $(p+1) \times p$ matrices obtained from $\mathcal{M}_p(A)$ and $\mathcal{M}_p(B)$ as follows.

By construction all the rows of $\mathcal{M}_p(A)$ and $\mathcal{M}_p(B)$ (other than the first) have exactly one fixed point. Let $\left\lceil \frac{p}{2} \right\rceil$ denote $\frac{p}{2}$ rounded to the next integer, and note that $2 \times \left\lceil \frac{p}{2} \right\rceil = p+1$.

The following table summarizes the fixed points for each row j (other than the first) of $\mathcal{M}_p(A)$ and $\mathcal{M}_p(B)$.

Row	Range of k	$\mathcal{M}_p(A)$	$\mathcal{M}_p(B)$
$j = 2k$	$1, \ldots, \left\lceil \frac{p}{2} \right\rceil$	$\left\lceil \frac{p}{2} \right\rceil - (k-1)$	$(p + \left\lceil \frac{p}{2} \right\rceil) - (k-1)$
$j = 2k+1$	$1, \ldots, \left\lceil \frac{p}{2} \right\rceil - 1$	$p - (k-1)$	$2p - (k-1)$

Then $\mathcal{M}'_p(A)$ and $\mathcal{M}'_p(B)$ are obtained by exchanging the fixed points of $\mathcal{M}_p(A)$ and $\mathcal{M}_p(B)$ row by row. For example, agent 1, which appears in entry $(p+1, 1)$ of $\mathcal{M}_p(A)$ remains unmatched in the last row of $\mathcal{M}_p(A)$ and agent $p+1$ which occupies entry $(p+1, 1)$ of $\mathcal{M}_p(B)$ remains unmatched in the last row of $\mathcal{M}_p(B)$. Then, the last row of $\mathcal{M}'_p(A)$ is obtained by replacing agent 1 in the last row of $\mathcal{M}_p(A)$ by agent $p+1$. Analogously, the

last row of $\mathcal{M}'_p(B)$ is obtained by substituting agent $p+1$ in the last row of $\mathcal{M}_p(B)$ by agent 1. A similar procedure is conducted to obtain the other rows of $\mathcal{M}'_p(A)$ and $\mathcal{M}'_p(B)$.

Now, with the help of Lemma 4.1, the reader can verify that the $2p \times 2p$ matrix

$$\mathcal{M}_{2p} = \left[\begin{array}{cc} \mathcal{M}'_p(A) & \mathcal{M}'_p(B) \\ L^-(B)_{-1} & L^+(A)_{-1} \end{array} \right]$$

is a maximal matching matrix for the population $X = \{1, \ldots, 2p\}$.

Next, suppose that a maximal matching matrix exists for a population $2n = 2^k p$, where $k \geq 1$. A procedure analogous to the one illustrated in Case 1 shows that a maximal matching matrix exists for $2n = 2^{k+1} p$. ∎

Theorem 4.2 shows the existence of maximal matching matrices for any finite population and in some cases provides algorithms of constructing maximal matching matrices. We illustrate this in the examples below.

Example 4.3. We start with the construction of a maximal matching matrix for an odd population with $n = 3$. Using the Latin square construction $\#3$ we obtain

$$\mathcal{M}_3 = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{array} \right].$$

Notice that this construction does not work with even populations. Indeed, the number n would appear in the last column of both the first and the third row, which is not a matching matrix for even populations. ∎

Example 4.4. Consider now an even population $X = \{1, 2, 3, 4, 5, 6\}$, i.e., we have $k = 1$ and $p = 3$. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Then

$$\mathcal{M}'_3(A) = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \\ 4 & 3 & 2 \end{array} \right] \quad \text{and} \quad \mathcal{M}'_3(B) = \left[\begin{array}{ccc} 4 & 5 & 6 \\ 6 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 6 & 5 \end{array} \right].$$

Also, notice that given the definitions of $L^+(A)$ and $L^-(B)$, we obtain the matrices

$$L^+(A)_{-1} = \left[\begin{array}{ccc} 3 & 1 & 2 \\ 2 & 3 & 1 \end{array} \right] \quad \text{and} \quad L^-(B)_{-1} = \left[\begin{array}{ccc} 5 & 6 & 4 \\ 6 & 4 & 5 \end{array} \right].$$

Thus, the 6×6 matrix

$$\mathcal{M}_6 = \begin{bmatrix} \mathcal{M}'_3(A) & \mathcal{M}'_3(B) \\ \\ L^-(B)_{-1} & L^+(A)_{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{bmatrix}$$

is a maximal matching matrix for the population X. ∎

When the population is of size $n = 2^k$ there are other useful ways of constructing maximal matching matrices. Indeed, we can always use the construction in the proof of Lemma 4.1 repeatedly or we can use another method that is described next.

5. Pairing agents in a population of size 2^k

In this section we consider a population X whose cardinality is a power of two, that is, $X = \{1, 2, \ldots, 2^k\}$ for some k. Again, we want to find out for how many periods we can pair every agent in X to everyone else exactly once, ensuring that all agents are paired in every period. This type of matching is of interest to experimental economists, where subjects should be handled parsimoniously. Clearly, there cannot be more than $2^k - 1$ rounds of matching since each agent $x \in X$ can meet at most $2^k - 1$ other agents. Therefore, let $t = 1, \ldots, 2^k - 1$ denote the number of matching periods. In Theorem 4.2 we showed that an absolute strangers matching protocol exists for this case. In what follows we will additionally show how to construct it. Our construction is recursive and takes k steps to be completed. We denote the arbitrary step of this construction by s.

In each step s we construct 2^{k-s} matching matrices of size $2^s \times 2^s$ labelled as

$$M_s^1, M_s^2, \ldots, M_s^{2^{k-s}}.$$

Each matrix M_s^i is a $2^s \times 2^s$ maximal matching matrix with the following 2^s symbols

$$\{2^s(i-1) + 1, \ldots, 2^s i\}.$$

For $s = 0$, we start by letting the matrices $M_0^1, M_0^2, \ldots, M_0^{2^k}$ be the 1×1 matrices:

$$M_0^1 = 1, M_0^2 = 2, \ldots, M_0^{2^k} = 2^k.$$

Now for the inductive procedure, if for some $s \geq 1$ the matching matrices

$$M_{s-1}^1, M_{s-1}^2, \ldots, M_{s-1}^{2^{k-s+1}}$$

have been constructed, then for each $1 \leq i \leq 2^{k-s}$ we let

$$M_s^i = \begin{bmatrix} M_{s-1}^{2i-1} & M_{s-1}^{2i} \\ \\ M_{s-1}^{2i} & M_{s-1}^{2i-1} \end{bmatrix}$$

Note that the second row of each M_s^i is obtained by shifting the matrices M_{s-1}^{2i-1} and M_{s-1}^{2i} in the first row, by one position to the left. That is, each M_s^i is a partitioned matrix whose blocks are the submatrices M_{s-1}^{2i-1} and M_{s-1}^{2i}.

Each matrix M_s^i is a matching matrix as we establish in the following.

Lemma 5.1. *If the population is of the form $X = \{1, \ldots, 2^k\}$, then for each $s = 1, \ldots, k$ and $i = 1, \ldots, 2^{k-s}$ the matrix M_s^i is a $2^s \times 2^s$ maximal matching matrix with symbols $\{(i-1)2^s + 1, \ldots, i2^s\}$.*

In particular, the recursively constructed matrix $M_k^1 = \mathcal{M}_{2^k}$ is a maximal matching matrix for the population X.

Proof. In order to show that M_k^1 identifies an absolute strangers matching protocol for the whole population X, we need to show that M_s^i is a matching matrix for each $s = 1, \ldots, k$ and each $i = 1, \ldots, 2^{k-s}$.

The proof proceeds by induction on s. Consider $s = 1$ and observe that for each $i = 1, \ldots, 2^{k-1}$, the matrix M_1^i is a matching matrix by construction since:

$$M_1^i = \begin{bmatrix} M_0^{2i-1} & M_0^{2i} \\ M_0^{2i} & M_0^{2i-1} \end{bmatrix} = \begin{bmatrix} 2i-1 & 2i \\ 2i & 2i-1 \end{bmatrix}$$

In particular, for all $i = 1, \ldots, 2^{k-1}$, each matrix M_1^i is a maximal matching matrix with symbols $\{(i-1)2 + 1, \ldots, i2\}$.

Now, assume that for all $i = 1, \ldots, 2^{k-s-1}$ the matrix M_{s-1}^i is a $2^{s-1} \times 2^{s-1}$ maximal matching matrix with symbols $\{(i-1)2^{s-1} + 1, \ldots, i2^{s-1}\}$. We need to show that for all $i = 1, \ldots, 2^{k-s}$ the matrix M_s^i is a $2^s \times 2^s$ maximal matching matrix with symbols $\{(i-1)2^s + 1, \ldots, i2^s\}$. Recall that

$$M_s^i = \begin{bmatrix} M_{s-1}^{2i-1} & M_{s-1}^{2i} \\ M_{s-1}^{2i} & M_{s-1}^{2i-1} \end{bmatrix}. \tag{5.1}$$

By the induction hypothesis, M_{s-1}^{2i-1} is a $2^{s-1} \times 2^{s-1}$ maximal matching matrix with symbols $\{(i-1)2^s + 1, \ldots, i2^s - 2^{s-1}\}$ and M_{s-1}^{2i} is a $2^{s-1} \times 2^{s-1}$ maximal matching matrix with symbols $\{i2^s - 2^{s-1} + 1, \ldots, i2^s\}$. Also, notice that

$$\{(i-1)2^s + 1, \ldots, i2^s - 2^{s-1}\} \cap \{i2^s - 2^{s-1} + 1, \ldots, i2^s\} = \emptyset,$$

i.e., the matrices M_{s-1}^{2i-1} and M_{s-1}^{2i} have distinct symbols. Therefore, by the construction provided in (5.1), we have that M_s^i is a $2^s \times 2^s$ maximal matching matrix with symbols $\{(i-1)2^s + 1, \ldots, i2^s\}$. Finally, note that the matrix obtained in the last step is a maximal matching matrix for the population, i.e., $M_k^1 = \mathcal{M}_{2^k}$. ■

Since each matching matrix M_s^i is of size 2^s, then it provides matches for 2^s distinct agents over the course of $2^s - 1$ periods. Therefore, the set $\{M_s^i : i = 1, \ldots, 2^{k-s}\}$ defined for $s = 1, \ldots, k-1$ provides a partition of the population in bilateral matches, up to period

$t = 2^s - 1$. When $s = k$ we have a single $2^k \times 2^k$ maximal matching matrix $M_k^1 = \mathcal{M}_{2^k}$ that gives us a matching of the whole population for a total of $2^k - 1$ periods.

Furthermore, this construction can be extended to obtain absolute strangers bilateral matching processes for countably infinite populations.[6]

Example 5.2. We illustrate our procedure by letting $k = 3$ and considering the population $X = \{1, \ldots, 8\}$. Lemma 5.1 indicates that we can bilaterally match agents as absolute strangers for $2^k - 1 = 7$ periods. We proceed as follows. First, start by defininig $M_0^i = i$ for $i = 1, \ldots, 8$. Then, the first matching is given by

$$\begin{bmatrix} M_1^1 & M_1^2 & M_1^3 & M_1^4 \end{bmatrix},$$

i.e., we have the four Latin squares of order 2×2

$$M_1^1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \; M_1^2 = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, \; M_1^3 = \begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}, \; M_1^4 = \begin{bmatrix} 7 & 8 \\ 8 & 7 \end{bmatrix}.$$

That is, agent 1 is paired to agent 2, agent 3 to 4, and so on and so forth.

Next, note that 1 has met 2 but he did not meet 3 and 4, and similarly 5 has met 6 but he has not met 7 and 8. When $s = 2$, the matching for the first three periods is given by $2^{3-2} = 2$ Latin squares of order $2^2 \times 2^2$

$$M_2^1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad M_2^2 = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 \\ 7 & 8 & 5 & 6 \\ 8 & 7 & 6 & 5 \end{bmatrix}.$$

That is, agent 1 is paired with agent 2, in period 1, with agent 3 in period 2 and with agent 4 in period 3.

Finally, when $s = k = 3$, we obtain a matching square of order $2^3 \times 2^3$ providing us with absolute strangers for $2^3 - 1 = 7$ periods

$$M_3^1 = \mathcal{M}_8 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

---

[6]Those matching matrices can be used also to construct matching processes where agents are more than absolute strangers. For instance, we can bilaterally match agents in every period t by selecting row $j = 1 + 2^{t-1}$ of the matching matrix. This is equivalent to a strongly anonymous matching process as formalized in [1].

For instance, in period $t = 3$, agent 4 meets agent 1, while in period $t = 7$ agent 4 meets agent 5. ■

6. A Practical Application to Experimental Economics

Consider the following setting for an economic experiment. We want to collect data on a repeated duopoly game involving two different types of subjects, say buyers and sellers. Suppose we wish to run the experiment for 40 periods and in each period we wish to match a buyer to a seller. Suppose also that we can only recruit 8 subjects, four of each type. Clearly, we can pair subjects so that every buyer meets every seller exactly once[7] for at most 4 periods. However, we would like to be as close as possible to an absolute strangers matching protocol. That is, we wish to minimize repeated interaction.[8] Randomizing equally over all possible matches would not allow us to achieve this goal, unfortunately. Indeed, the probability that agents are absolute strangers under a random matching protocol is very low.

To see why, start by noticing that the number of all possible matchings (with repetition) is $(n!)^n$. That is, we must consider all matrices in which the symbols $1, \ldots, n$ appear exactly once in every row, but can appear more than once in a column. Thus, we have $n!$ choices for the first row, $n!$ choices for the second row, and so on until $n!$ choices for the nth row.

This implies that, if we assume that all matchings are equally likely, the probability of obtaining an absolute strangers matching is given by

$$p_n = \frac{\ell_n}{(n!)^n}.$$

Here, ℓ_n is the number of Latin squares that can be created for a population of size n. For example, if $n = 3$ then $\ell_3 = 18$ and so the probability that agents are absolute strangers is $p_3 = \frac{1}{18}$. For $n = 4$ we have $\ell_4 = 576$ and so the probability is $p_4 = \frac{1}{576}$. For numbers that can be calculated, greater values of n are associated to progressively smaller probabilities (e.g, $p_{10} = 2.5212 \times 10^{-29}$). That is to say, if matchings are selected randomly, the probability of selecting an absolute strangers matching becomes very small as n increases.

This suggests that knowing how to construct absolute strangers matching protocols may be important, since it allows us to construct several *distinct* matching matrices, i.e., several distinct absolute strangers matching protocols. This allows us flexibility in the design of the matching protocol and, in particular, it allows us to decrease the chance that the same partition of the population is repeated.

For example, given our population of size four we will constract several Latin squares of order 4 and then randomly select matchings from them. We need more than one matching

---

[7]And nobody is unmatched.

[8]Strangers matching protocols are used in experimental economics to reduce the impact of repeated game effects while allowing for learning over time.

square because using one only implies that if a match is repeated, then the partition of the entire population is also repeated. This is comparable to having a "periodic" repeated game, which may be desirable to avoid.

For a concrete example, suppose we want to match buyers $A = \{a, b, c, d\}$ with sellers $B = \{1, 2, 3, 4\}$. Generate several distinct Latin squares (we can generate up to 4!3!4), and use them to form matching matrices. Note that distinct Latin squares generate distinct matching matrices. For example, it is easy to see that the Latin square

$$
L^1 = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
$$

gives rise to the matching matrix

$$
\mathrm{M}(A, B) = \begin{bmatrix}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\
1 & 2 & 3 & 4 & a & b & c & d \\
2 & 3 & 4 & 1 & d & a & b & c \\
3 & 4 & 1 & 2 & c & d & a & b \\
4 & 1 & 2 & 3 & b & c & d & a
\end{bmatrix}.
$$

Similarly, the following Latin squares generate matching matrices different than M(A,B).

$$
L^2 = \begin{bmatrix}
2 & 1 & 4 & 3 \\
3 & 2 & 1 & 4 \\
4 & 3 & 2 & 1 \\
1 & 4 & 3 & 2
\end{bmatrix}, \quad
L^3 = \begin{bmatrix}
3 & 1 & 4 & 2 \\
4 & 2 & 3 & 2 \\
1 & 3 & 2 & 4 \\
2 & 4 & 1 & 3
\end{bmatrix}
$$

$$
L^4 = \begin{bmatrix}
4 & 1 & 2 & 3 \\
1 & 3 & 4 & 2 \\
2 & 4 & 3 & 1 \\
3 & 2 & 1 & 4
\end{bmatrix}, \quad
L^5 = \begin{bmatrix}
1 & 2 & 4 & 3 \\
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4 \\
4 & 3 & 1 & 2
\end{bmatrix}
$$

Next, we can select a matching matrix every four periods, randomly and independently, to minimize repeated interaction while preserving randomness in matching *as well as* in partitioning the population. If we were to use only L^1 and repeat it over time, for instance, if a is matched repeatedly to 1, then the matches $(b, 2), (c, 3), (d, 4)$ are also repeated. Thus, knowing how to construct matching matrices can allow to generate a matching process such that we disentangle information about composition of a subject's match (presumably observable only in the match) from info about entire partition (presumably unobservable).

7. CONCLUSION

We have offered a procedure to match bilaterally agents in a finite population, so that in each period everybody has a partner (but at most one) and meets everybody else exactly once. Our method is simple and makes use of the mathematics of Latin squares. It can have practical applications in the construction of decentralized trading environments that are often employed in both theoretical and experimental economics to model trading or informational frictions. The basic procedure to create these type of meetings is simple, can be done with pencil and paper, and can be extended to countably infinite population by means of a straighforward recursive process.

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