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Abstract

In the Colonel Blotto game, each of two players simultaneously allocates his fixed budget of a resource across a finite number n of battlefields. Within each battlefield, the player that allocates the higher level of the resource wins the battlefield. Each player's payoff is equal to the sum of the values of the battlefields he wins. In this paper we examine a multi-dimensional incomplete information version of the Colonel Blotto game in which each player's n-tuple of battlefield valuations is drawn from a common n-variate joint distribution function.

JEL Classification: C72, D72, D74

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1 Introduction

The classic formulation of the Colonel Blotto game is a two-player constant-sum game with complete information (Borel 1921). There are a finite number n of battlefields. Each player simultaneously allocates his fixed amount of a resource across the n battlefields. Within each battlefield, the player that allocates the higher level of the resource wins the battlefield, and each player's payoff is equal to the number of battlefields won.[1] A distinguishing feature of the Colonel Blotto game is that, except for trivial cases, an equilibrium is a pair of non-degenerate multi-dimensional mixed strategies.

This note investigates a multi-dimensional incomplete information version of the Colonel Blotto game with two risk-neutral players. The incomplete information is multi-dimensional in that each player's n-tuple of battlefield valuations is drawn from a common n-variate joint distribution. Our game is therefore within the class of games with multi-dimensional incomplete information in which each player has a multi-dimensional action space. In the case in which the joint distribution of battlefield valuations is a uniform distribution over the nonnegative points on the surface of a sphere, we show that there exists an intuitive symmetric pure-strategy Bayesian equilibrium.

Most closely related to our focus is Adamo and Matros (2009) who introduce incomplete information in the Colonel Blotto game framework. In that model, k players compete across a set of n battlefields with valuations that are common knowledge. These valuations may vary across battlefields, but each battlefield's valuation is the same for all players. The incomplete information is in the form of uncertainty regarding the players' respective budgets of the resource, which are assumed to be independent draws from a common distribution that is absolutely continuous on the closed interval between 0 and the sum of the individual battlefield valuations. Under the added assumption that the distribution of the highest

[1] For further details on this class of games, see Roberson (2006) and Kovenock and Roberson (2010).

order statistic of the other $k-1$ of the players' budgets is concave, there exists a monotonic symmetric pure-strategy Bayesian equilibrium of that model in which each player allocates to each battlefield a share of his budget equal to that battlefield's value divided by the sum of all battlefield valuations. That is, the one-dimensional nature of the incomplete information allows for each element in the n-tuple of a player's allocation of force across battlefields to be conditioned on a single variable, his budget.

Also related are Che and Gale (1998) and Pitchik (2009) who examine incomplete information auctions with budget-constrained players. In fact, our model corresponds to the case of two budget-constrained players competing across a set of all-pay auctions with incomplete information with the exception that for each player any unused resources are forfeited. The case of a single all-pay auction with incomplete information is addressed by Amann and Leininger (1996), who examine asymmetrically distributed independent private values and Krishna and Morgan (1997), who examine symmetrically distributed affiliated signals. In this note we show that if the joint distribution of the players' valuations takes a particular parametric form, then the analysis of the all-pay auction with incomplete information can be extended to cover our case of multiple auctions, binding budget constraints, and use-it-or-lose-it resources.

2 Model

There are two risk-neutral players, $i \in \{A, B\}$, and a finite number n of battlefields, $j \in \{1, \ldots, n\}$. Each player has a budget of one unit of a homogenous resource. Let $\mathcal{B}$ denote the set of feasible allocations, or n-tuples of bids,

$$\mathcal{B} \equiv \left\{ \mathbf{b} \in \mathbb{R}^n_+ \middle| \sum_{j=1}^{n} b_j \leq 1 \right\}.$$

For the sake of brevity, we focus on the case that the number of battlefields, n, is equal to three. However, this approach readily generalizes to any n that is an integer multiple of three.[2]

The two players simultaneously allocate resources across the three battlefields. In each battlefield the player that allocates the higher level of the resource wins. If the players allocate the same level of the resource to a given battlefield, each player wins with equal probability. Each player's payoff equals the sum of the valuations of the battlefields won.

Each player's 3-tuple of battlefield valuations, $\mathbf{v} = (v_1, v_2, v_3)$, is private information and is assumed to be independently drawn from a common 3-variate distribution function. In general, our problem gives rise to significant technical challenges. However, for the following parametric specification of the joint distribution of the players' types, there exists an intuitive pure-strategy Bayesian equilibrium.[3]

Consider the sphere of unit radius centered at the origin in $\mathbb{R}^3$, and let $\mathcal{S}$ denote the set of points on the surface of this unit sphere that lie in $\mathbb{R}^3_+$:

$$\mathcal{S} \equiv \left\{ \mathbf{v} \in \mathbb{R}^3_+ \middle| \sum_{j=1}^{3} (v_j)^2 = 1 \right\}.$$

Let $\widehat{P}$ denote the 3-variate distribution function formed by uniformly distributing mass over $\mathcal{S}$. For each battlefield $j \in \{1, 2, 3\}$, the univariate distribution function of v_j, $F_j(z)$, is equal to the $\widehat{P}$-volume of the set of $\mathbf{v} \in \mathcal{S}$ such that $v_j \leq z \leq 1$.

[2] Just as in the classic complete information version of the Colonel Blotto game, moving from two to three battlefields allows for much greater flexibility in the choice of a relevant joint distribution (Roberson 2006).

[3] Although we focus on the case in which the support of the joint distribution of battlefield valuations lies on one particular surface, this approach can be extended to other surfaces.

3 Results

A strategy is a function $\mathbf{b} : \mathcal{S} \to \mathcal{B}$. Denote player i's 3-tuple of bids as $\mathbf{b}^i(\{v^i_j\}^3_{j=1}) = (b^i_1(\{v^i_j\}^3_{j=1}), b^i_2(\{v^i_j\}^3_{j=1}), b^i_3(\{v^i_j\}^3_{j=1}))$.

Proposition 1. *If each player's n-tuple of battlefield valuations is drawn from the joint distribution $\widehat{P}$, then*

$$\boldsymbol{b}^*(\{v_j\}^3_{j=1}) = \left\{(v_1)^2, (v_2)^2, (v_3)^2\right\} \tag{1}$$

is a symmetric pure-strategy Bayesian equilibrium.

We briefly outline the proof of Proposition 1. Lemma 1 states a useful property of the joint distribution $\widehat{P}$.

Lemma 1. *The joint distribution $\widehat{P}$ satisfies the property that for each $j \in \{1, \ldots, 3\}$, $F_j(v_j) = v_j$ for $v_j \in [0, 1]$, i.e. each v_j is distributed uniformly on $[0, 1]$.*

Lemma 1 follows from the properties of spherical segments. In particular, note that the surface area of any spherical segment is $2\pi rh$ where h is the height between the two parallel planes that cut the sphere. The unit sphere has radius $r = 1$, surface area 4π, and exactly (1/8)th of the sphere is contained in the positive orthant of $\mathbb{R}^3$. Thus, the surface area of the relevant portion of the unit sphere is $(\pi/2)$. Similarly, the surface area of the spherical segment formed by cutting the surface $\mathcal{S}$ with two parallel planes is $(\pi h/2)$. The probability that $v_1 \leq z \leq 1$ is equal to the probability that v_1 is contained in the spherical segment formed by cutting the surface $\mathcal{S}$ with the v_2, v_3-plane and a parallel plane which is height z above the v_2, v_3-plane. This probability is equal to the ratio of the relevant surface area of this spherical segment and the relevant surface area of the sphere $(2\pi z)/(2\pi) = z$ for all $z \in [0, 1]$. A similar argument applies for the v_2 and v_3 univariate marginal distributions.

Note that $\mathbf{b}^*(\{v_j\}^3_{j=1})$ is an admissible bidding strategy for all $\mathbf{v} \in \mathcal{S}$. Then, because player B is following the equilibrium strategy $\mathbf{b}^*(\{v^B_j\}^3_{j=1})$, player A's optimization problem may be written as

$$\max_{\{\mathbf{b}^A \in \mathcal{B}\}} \sum_{j=1}^{3} v_j^A \Pr(b_j^A > b_j^*(\{v_j^B\}_{j=1}^3)) \tag{2}$$

where for any $\mathbf{b}^A \in \mathcal{B}$ and any $j \in \{1, 2, 3\}$,

$$\Pr(b_j^A > b_j^*(\{v_j^B\}_{j=1}^3)) = F_j\left(\sqrt{b_j^A}\right) = \sqrt{b_j^A}.$$

From equation (2), the Lagrangian for player A's optimization problem may be written as

$$\max_{\{\mathbf{b}^A \in \mathcal{B}\}} \sum_{j=1}^{3} \left[v_j^A \sqrt{b_j^A} - \lambda_A b_j^A\right] + \lambda_A. \tag{3}$$

Solving the f.o.c. for each battlefield j, $b_j^A = (v_j^A/(2\lambda_A))^2$. Recalling that in any optimal strategy $\sum_{j=1}^3 b_j^A = 1$ and $\sum_{j=1}^3 (v_j^A)^2 = 1$ for all $\mathbf{v}^A \in \mathcal{S}$, it follows that $\lambda_A = \frac{1}{2}$. Lastly, note that from (2) and (3) the expected payoff is an increasing concave function of $\mathbf{b}^A$ on $\mathcal{B}$. This completes the proof that the strategy $\mathbf{b}^*(\{v_j^A\}_{j=1}^3)$ is a globally optimal response for player A given that player B is using $\mathbf{b}^*(\{v_j^B\}_{j=1}^3)$. The case of player B follows directly.

The combination of multi-dimensional incomplete information and multi-dimensional action spaces usually entails significant technical difficulties. However, in the context of a Blotto game with multi-dimensional incomplete information we have shown that if the common joint distribution of battlefield valuations is uniformly distributed on the surface $\mathcal{S}$, then there exists an intuitive symmetric equilibrium $\mathbf{b}^*(\{v_j\}_{j=1}^3)$ in which for each battlefield j the coordinate function $b_j^*(\{v_{j'}\}_{j'=1}^3)$ is a function of only that battlefield's valuation v_j and is strictly increasing in that valuation.

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