Hotelling's Main Street Model: Undercut-Proof Equilibrium as a Constrained Optimization Problem

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Abstract

The paper analyzes a two-stage Main Street model. In the second stage, taking locations as given, each of two firms sets price to maximize own profit, subject to the constraint that it is not profitable for the other firm to undercut its price at its location. We find constrained price best-response equations and pure-strategy equilibrium prices for all pairs of locations. In the first stage, firms noncooperatively pick locations to maximize second-stage payoffs. We find location best-response equations and equilibrium locations. Equilibrium locations are efficient in the sense of minimizing transportation cost.

JEL categories: C72, D21, D43, L13

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1 Introduction

Hotelling's Main Street model is one of industrial organization's workhorse models of imperfect competition. Consumers are uniformly distributed along a line of length l. Two firms, firm A at distance a from the left end of the line and firm B at distance b from the right end of the line, supply a product that is differentiated by the distance from a consumer to a producing firm. Production cost is constant per unit of output, and normalized to be 0. Transportation cost is constant per unit distance. "Transportation cost" may be interpreted either literally or figuratively, in the latter case with locations on the line representing product characteristics. Each consumer purchases one unit of the product from the firm that offers the lowest delivered price — price at the supplying firm's location plus transportation cost.

d'Aspremont *et al.* (1979) formalize Hotelling's model as a two-stage game, with firms picking locations in the first stage and setting prices in the second. They show that there is no pure-strategy equilibrium in prices in the second stage if firms are "too close" to each other, because in that case one firm or the other or both would find it profitable to undercut its rival's price at the rival's location and supply the whole line.[1] Lacking second-stage pure-strategy equilibrium prices rules out first-stage analysis of noncooperative choices of locations and invalidates the Principle of Minimum Differentiation, which would have firms locate in the center of the line.[2,3]

The spatial economics literature is replete with modifications of the basic Main Street model that seek to deal with the non-existence of pure strategy equilibrium prices in the second stage of the Hotelling model. Some parts of Hotelling (1929) can be read as suggesting that he simply assumed that sales to a firm's hinterland customers were reserved for it.[4] Eaton (1972) introduces the concept of "modified zero conjectural variations," that "each

[1] In the case of symmetric locations, they show that "too close" means $a = b > \frac{1}{4}$ (and give as well conditions that characterize the nonsymmetric case).

[2] The Principle of Minimum Differentiation is due to Chamberlin (1933), and was baptized by Boulding (1966).

[3] Osborne and Pitchik (1987) examine equilibrium choice of locations based on numerical analysis of mixed-strategy equilibrium prices, and find that for the symmetric case firms locate $0.27l$ from the ends of the line (that is to say, inside the quartile points identified by d'Aspremont *et al.*

[4] Other parts of the 1929 paper are not consistent with this interpretation; see Hinloopen and Martin (2024) for discussion.

producer in setting his price and location assumes that the other will remain in the same location, charging the same price, subject to the qualification that the action of one producer does not completely eliminate the other," thus "rul[ing] out the possibility that A will undersell B to the left of B or that B will undersell A to the right of A."

Shy (2002) defines the undercut-proof property, which he later explains (Shy, 2024, p. 650) as

> In an [undercut-proof equilibrium], each firm maximizes its price subject to the constraint that no other firm would find it profitable to undercut its own price and grab its market share.

Undercut-proofness is the focus of this paper, which is organized as follows. Section 2 gives a brief review of previous work most closely related to the present paper.[5] Section 3 formalizes the constrained optimization problem of one of the firms,[6] and Section 4 presents the price best-response functions. Section 5 gives possible second-period equilibrium prices and payoffs, and Section 6 shows which payoffs hold in different regions of location space. Section 7 gives first-period location best responses and equilibrium locations; Section 8 concludes.

2 Literature

Hamilton *et al.* (1991) examine a two-stage game *à la* Hotelling with the requirement that second-stage prices result in allocations of the product that are in the core, *i.e.* (1991, pp. 927-928) "no group of consumers can join together with a firm and benefit from a change in the announced price schedule." This incorporates the central idea of no-undercutting equilibrium (1991, p. 931): "An allocation is in the core if and only if it is supported by a price pair with the property that no firm can increase its payoff by a unilateral price reduction." The equilibrium locations obtained here using best response functions are identical to their results for the case that firms do not observe customers' locations.

[5] Any attempt at a complete review of this large literature would be out of place here. For a more extensive (but still incomplete) discussion, see Hinloopen and Martin (2024, Section 2).

[6] The solution is outlined in Appendix B. Full details are contained in an unpublished Appendix that is available on request from the author.

Iskakov and Iskakov (2012) analyze the nature of equilibrium in secure strategies, due to Iskakov (2005), in the Hotelling model. Equilibrium in secure strategies requires players "to maximize their profit under the condition of security against the actions of other players." In the context of the Hotelling model, the action to be secured against is "being driven out of the market by [price] undercutting." They note that the undercut-proof equilibrium of Shy (2002) is a precursor of equilibrium in secure strategies and that the two concepts coincide in a version of the Hotelling line with a discrete distribution of consumers. The present paper shows that the location equilibria of the two concepts coincide with the uniform continuous distribution of consumers assumed by Hotelling.

The paper in the literature that is most closely related to this one is Shy (2024), who presents an algorithm to work out what in my framework are second-stage outcomes for arbitrary pairs of locations on the Hotelling line. The contributions of this paper are to derive second-stage outcomes using constrained price best-response functions and, based on results for the second stage, to find location best-response functions and pure-strategy equilibrium locations for the first stage.

3 Firm A's constrained price-setting problem

Without loss of generality (see Appendix A), we normalize the length of the Hotelling line and transportation cost per unit distance to be 1. In this normalized specification of the Main Street model, (Figure 1) firm A is located a distance α from the left end of the line, and firm B is located a distance β from the right end of the line. Firm A is located to the left of firm B.[7]

If firms share the middle of the line, demand equations and payoff functions are

$$q_A = \frac{1}{2}\left(1 + \alpha - \beta + p_B - p_A\right) \tag{1}$$

$$q_B = \frac{1}{2}\left(1 - \alpha + \beta + p_A - p_B\right) \tag{2}$$

and

$$\pi_A = \frac{1}{2} p_A \left(1 + \alpha - \beta + p_B - p_A\right) \tag{3}$$

[7] The normalized specification is used throughout the paper.

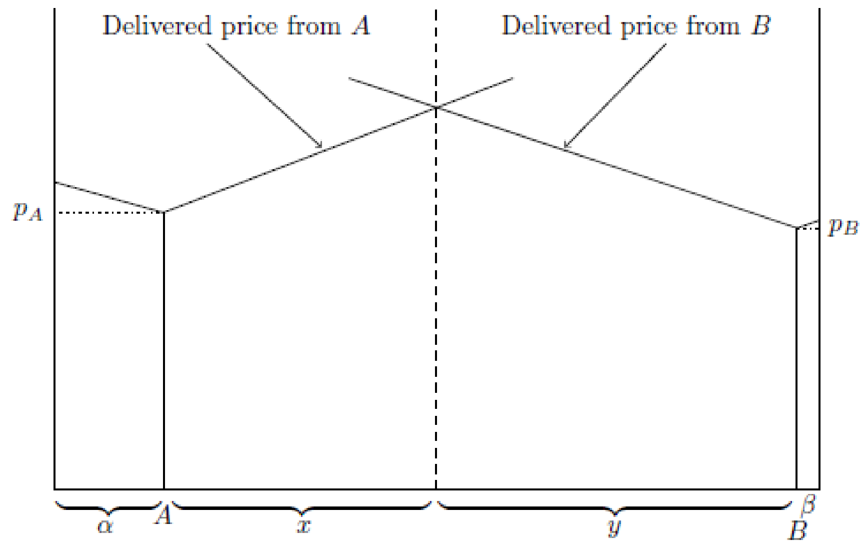

Figure 1: Hotelling linear duopoly model; $\alpha = 4/35$, $\beta = 1/35$, $t = 1$, length of line normalized to 1. Based on Hotelling (1929, Figure 1).

$$\pi_B = \frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right), \tag{4}$$

respectively.

If firm B sets a price slightly below $p_A - (1 - \alpha - \beta)$, it undercuts A's price at A's location, supplies whole line, and gets payoff just below

$$p_A - (1 - \alpha - \beta). \tag{5}$$

A's undercut-proof constrained optimization problem is then to maximize its payoff subject to the constraint that the choice of p_A makes it weakly unprofitable for B to undercut A's price at A's location,[8,9]

$$\max_{p_A \geq 0} \frac{1}{2} p_A \left(1 + \alpha - \beta + p_B - p_A\right)$$

$$\ni: \frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right) \geq p_A - (1 - \alpha - \beta). \tag{6}$$

[8]Because firm A's constraint depends on firm B's choice variable, this is a Generalized Nash Equilibrium Problem. See Facchinei and Kanzow (2010) for a survey.

[9]Shy (2024, p. 655) refers to the constraint as the global undercut-proof property.

A Lagrangian for this constrained optimization problem is[10]

$$\max_{p_A \geq 0} \mathcal{L}_A = \frac{1}{2} p_A \left(1 + \alpha - \beta + p_B - p_A\right)$$

$$+ \lambda_A \left[\frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right) - p_A + \left(1 - \alpha - \beta\right)\right]. \tag{7}$$

The Karush-Kuhn-Tucker (KKT) conditions that characterize the solution of (7) are

$$\frac{\partial \mathcal{L}_A}{\partial p_A} = \frac{1}{2}\left(1 + \alpha - \beta + p_B - 2p_A\right) + \lambda_A \left(\frac{1}{2} p_B - 1\right) \leq 0 \tag{8}$$

$$\left[\frac{1}{2}\left(1 + \alpha - \beta + p_B - 2p_A\right) + \lambda_A \left(\frac{1}{2} p_B - 1\right)\right] p_A = 0 \tag{9}$$

$$p_A \geq 0. \tag{10}$$

$$\frac{\partial \mathcal{L}_A}{\partial \lambda_A} = \frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right) - p_A + \left(1 - \alpha - \beta\right) \geq 0 \tag{11}$$

$$\left[\frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right) - p_A + \left(1 - \alpha - \beta\right)\right] \lambda_A = 0 \tag{12}$$

$$\lambda_A \geq 0. \tag{13}$$

4 Price best-response functions

We characterize outcomes in the second stage of the game in two Lemmas.

The following price values, which depend on firms' locations, appear in the expressions for the price best-response functions:

$$p_B^T = \frac{1}{2}\left(1 - \alpha + \beta\right) + \sqrt{2\left(1 - 3\alpha - \beta\right) + \frac{1}{4}\left(1 - \alpha + \beta\right)^2} \tag{14}$$

$$p_B^U = \frac{1}{2}\left(1 - \alpha + \beta\right) + \sqrt{2\left(1 - \alpha - \beta\right) + \frac{1}{4}\left(1 - \alpha + \beta\right)^2} \tag{15}$$

[10] If p_A is sufficiently low, $p_A \leq 1 - \alpha - \beta$, even $p_B = 0$ would not undercut A's price at A's location. One could therefore take the range of p_A in (7) as $p_A \geq 1 - \alpha - \beta$. The solution to (7) subsumes results for this alternative formulation..

$$p_A^T = \frac{1}{2}(1+\alpha-\beta) + \sqrt{2(1-\alpha-3\beta) + \frac{1}{4}(1+\alpha-\beta)^2} \quad (16)$$

$$p_A^U = \frac{1}{2}(1+\alpha-\beta) + \sqrt{2(1-\alpha-\beta) + \frac{1}{4}(1+\alpha-\beta)^2}. \quad (17)$$

In the location-space triangle $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $\alpha + \beta \leq 1$, a firm's price best-response function has one of two possible configurations. If firm A is not "too close" to the center of the line, in a way that depends on firm B's location and is made precise in Lemma 1, the no-undercutting constraint does not bind A's best response for low values of p_B, but is binding for high values of p_B. For points in this part of the location triangle, p_B^T is the threshold value of p_B above which the no-undercutting constraint becomes binding on A. If the no-undercutting constraint binds, A sets a price below its unconstrained best response, to keep B's undercutting payoff identical to its no-undercutting payoff. As p_B rises above p_B^T, A's constrained best-response price continues to fall. p_B^U is the value of p_B that makes A's constrained best-response price equal to zero.

Outside the region of the location triangle where A's best-response function has two segments, the no-undercutting constraint is binding on A for all values of p_B.

p_A^T and p_A^U have similar interpretations for firm B's constrained optimization problem.

Lemma 1 (Price best-response functions) *In the location-space triangle* $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $\alpha + \beta \leq 1$

(a1) For

$$D_{A1} = 2(1-3\alpha-\beta) + \frac{1}{4}(1-\alpha+\beta)^2 \geq 0, \quad (18)$$

or equivalently

$$\alpha \leq 13 + \beta - 4\sqrt{2(5+\beta)}, \quad (19)$$

the equation of A's constrained price best-response function is

$$p_A = \begin{cases} \frac{1}{2}(1+\alpha-\beta+p_B) & 0 \leq p_B \leq p_B^T \\ \frac{2(1-\alpha-\beta)+(1-\alpha+\beta)p_B-p_B^2}{2-p_B} & p_B^T < p_B \leq p_B^U \end{cases}; \quad (20)$$

(a2) for $D_{A1} < 0$, or equivalently

$$\alpha > 13 + \beta - 4\sqrt{2(5+\beta)}, \quad (21)$$

the equation of A's price best-response function is

$$p_A = \frac{2(1-\alpha-\beta) + p_B(1-\alpha+\beta) - p_B^2}{2-p_B}, 0 \le p_B \le p_B^U. \tag{22}$$

(b1) For

$$D_{B1} = 2(1-\alpha-3\beta) + \frac{1}{4}(1+\alpha-\beta)^2 \ge 0, \tag{23}$$

or equivalently

$$\beta \le 13 + \alpha - 4\sqrt{2(5+\alpha)} \tag{24}$$

the equation of B's best-response function is

$$p_B = \begin{cases} \frac{1}{2}(1-\alpha+\beta+p_A) & 0 \le p_A \le p_A^T \\ \frac{2(1-\alpha-\beta)+(1+\alpha-\beta)p_A - p_A^2}{2-p_A} & p_A^T < p_A \le p_A^U \end{cases}; \tag{25}$$

(b2) for $D_{B1} < 0$ or equivalently

$$\beta > 13 + \alpha - 4\sqrt{2(5+\alpha)}, \tag{26}$$

the equation of B's best-response function is

$$p_B = \frac{2(1-\alpha-\beta) + (1+\alpha-\beta)p_A - p_A^2}{2-p_A}, 0 \le p_A \le p_A^U. \tag{27}$$

Proof: see Appendix B.

Note from the best-response equations that prices must be strictly less than 2.

The inequalities (19) and (24) divide the location space triangle into four regions, according to the shapes of firms' price best-response functions (Figure 2):

- Region *I*: both firms' price best-response functions have a lower portion where the no-undercutting constraint is not binding, and an upper portion where it is binding.
- Region *II*: A's price best-response function as in *I*, B's price best-response function is subject to the constraint throughout.
- Region *III*: B's price best-response function as in *I*, A's price best-response function is subject to the constraint throughout.

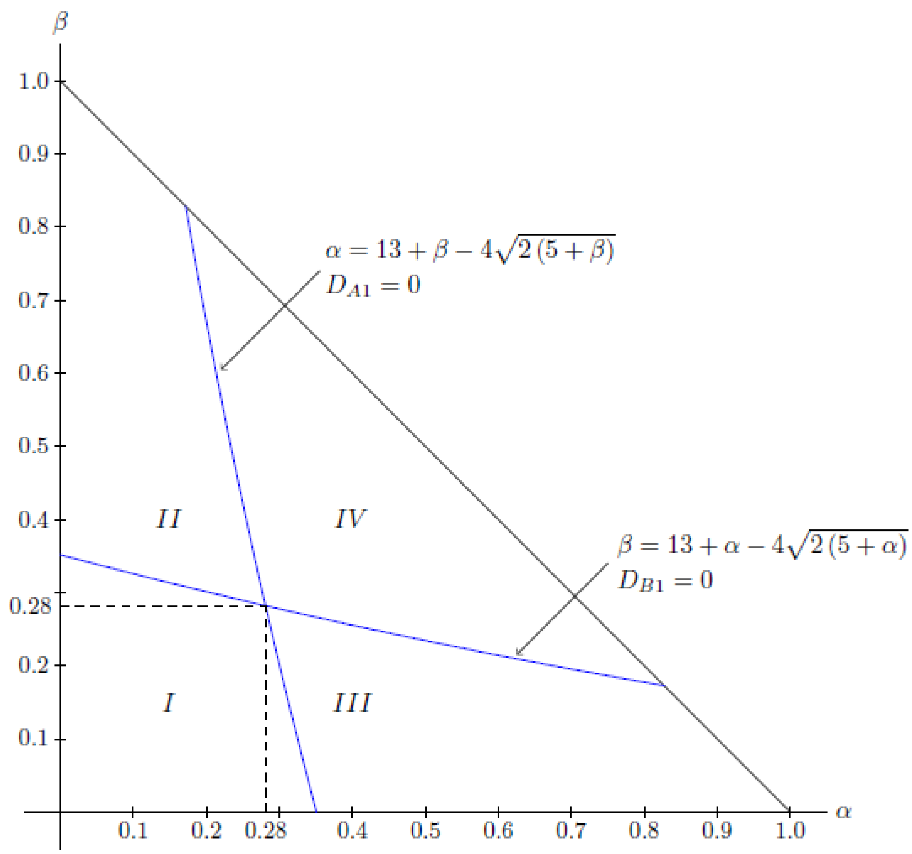

Figure 2: Combinations of best-response functions, by regions in location space.

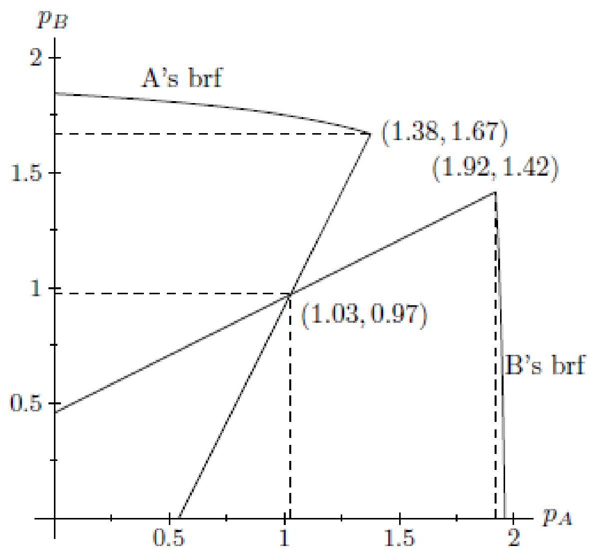

Figure 3: Price best-response functions, locations in Region I: $\alpha = 4/35$, $\beta = 1/35$ (Hotelling's first example, normalized).

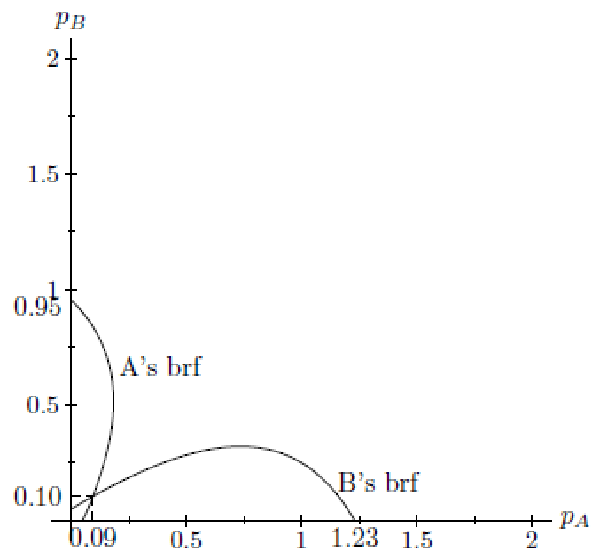

Figure 4: Price best-response functions, locations in Region IV: $\alpha = 11/20, \beta = 8/20$ (Hotelling's second example, normalized).

- Region IV: both firms' price best-response function are subject to the no-undercutting constraint throughout.

Figures 3 and 4 show best-response function configurations for two pairs of locations, both of which considered by Hotelling (1929). Figure 3 shows typical price-best response functions for region I: when firms are far enough apart, the no-undercutting constraint is not binding for low values of the other firm's price, but comes into play for high values. Figure 4 shows price best-response functions for region IV; firms are sufficiently close together that the no-undercutting constraint always determines best responses.[11] Other combinations of price best-response functions in different regions of the location triangle are shown below.

[11] These locations are labelled H_1 and H_2, respectively, in Figure 7.

5 Combinations of equilibrium prices and payoffs

Lemma 1 speaks to the shape of firms' best response functions at different points in location space. To show how equilibrium prices and payoffs differ by location, we first solve the alternative combinations of first-order conditions for the constrained optimization problem.

5.1 Both firms' best-response prices unconstrained

This is the case considered by Hotelling (1929). Best-response prices are given by the upper rows of (20) and (25). Equilibrium prices and payoffs are[12]

$$p_A^{uu} = 1 + \frac{\alpha - \beta}{3} \qquad p_B^{uu} = 1 + \frac{\beta - \alpha}{3} \tag{28}$$

and

$$\pi_A^{uu} = \frac{1}{2}\left(1 + \frac{\alpha - \beta}{3}\right)^2 \qquad \pi_B^{uu} = \frac{1}{2}\left(1 + \frac{\beta - \alpha}{3}\right)^2. \tag{29}$$

For symmetric locations ($\alpha = \beta$), equilibrium prices equal 1, each firm supplies half the line, and equilibrium profit per firm is 1/2.

5.2 A unconstrained, B constrained

Firm A is on the unconstrained portion of its best-response function, while firm B's best-response price is subject to the no-undercutting constraint throughout. Equilibrium prices and payoffs are

$$p_A^{uc} = 2\left(1 - \sqrt{\beta}\right) \qquad p_B^{uc} = 3 + \beta - \alpha - 4\sqrt{\beta} \tag{30}$$

and

$$\pi_A^{uc} = 2\left(1 - \sqrt{\beta}\right)^2 \qquad \pi_B^{uc} = \left(3 + \beta - \alpha - 4\sqrt{\beta}\right)\sqrt{\beta}. \tag{31}$$

[12] The superscript "uu" indicates that the value is for an equilibrium in which firm A (first letter) is unconstrained and firm B (second letter) is unconstrained. "uc" denotes an equilibrium in which firm A is unconstrained, firm B is constrained, and so on.

5.3 A constrained, B unconstrained

This is the mirror-image of the previous case.

$$p_A^{cu} = 3 + \alpha - \beta - 4\sqrt{\alpha} \qquad p_B^{cu} = 2\left(1 - \sqrt{\alpha}\right) \tag{32}$$

$$\pi_A^{cu} = \left(3 + \alpha - \beta - 4\sqrt{\alpha}\right)\sqrt{\alpha} \qquad \pi_B^{cu} = 2\left(1 - \sqrt{\alpha}\right)^2. \tag{33}$$

5.4 Both firms' best response prices constrained

Best-response price functions are (22) and (27). This system of equations can be reduced to a pair of cubic equations, one for each price:

$$p_A^3 - (5 - \alpha - 3\beta)\, p_A^2 + 2\left(4 - 2\alpha - 3\beta + \beta^2 - \alpha\beta\right) p_A - 4\,(1 + \beta)\,(1 - \alpha - \beta) = 0 \tag{34}$$

and

$$p_B^3 - (5 - 3\alpha - \beta)\, p_B^2 + 2\left(4 - 3\alpha - 2\beta + \alpha^2 - \alpha\beta\right) p_B - 4\,(1 + \alpha)\,(1 - \alpha - \beta) = 0. \tag{35}$$

Since the constraints bind, payoffs for this case satisfy

$$\pi_A^{cc} \equiv p_B^{cc} - (1 - \alpha - \beta) \qquad \pi_B^{cc} \equiv p_A^{cc} - (1 - \alpha - \beta). \tag{36}$$

(34) and (35) yield analytic expressions for p_A and p_B, respectively (Iskakov and Iskakov, 2012, Theorem 3). From (36), however, what is required to analyze the first-stage location choice game is an analytic result for the sign of $\frac{\partial \pi_A^{cc}}{\partial \alpha} = \frac{\partial p_B^{cc}}{\partial \alpha} + 1$, and this is not forthcoming. When we turn to consideration of the first-stage game, therefore, we rely on numerical analysis of A's payoffs if the no-undercutting constraint holds for both firms.

6 Equilibrium prices and payoffs by region in location space

For equilibrium prices to have both firms unconstrained requires that each firm's equilibrium price be less than or equal to the threshold value for the other firm's best-response equation. That is, equilibrium prices must satisfy

$$p_A^{uu} \le p_A^T, \tag{37}$$

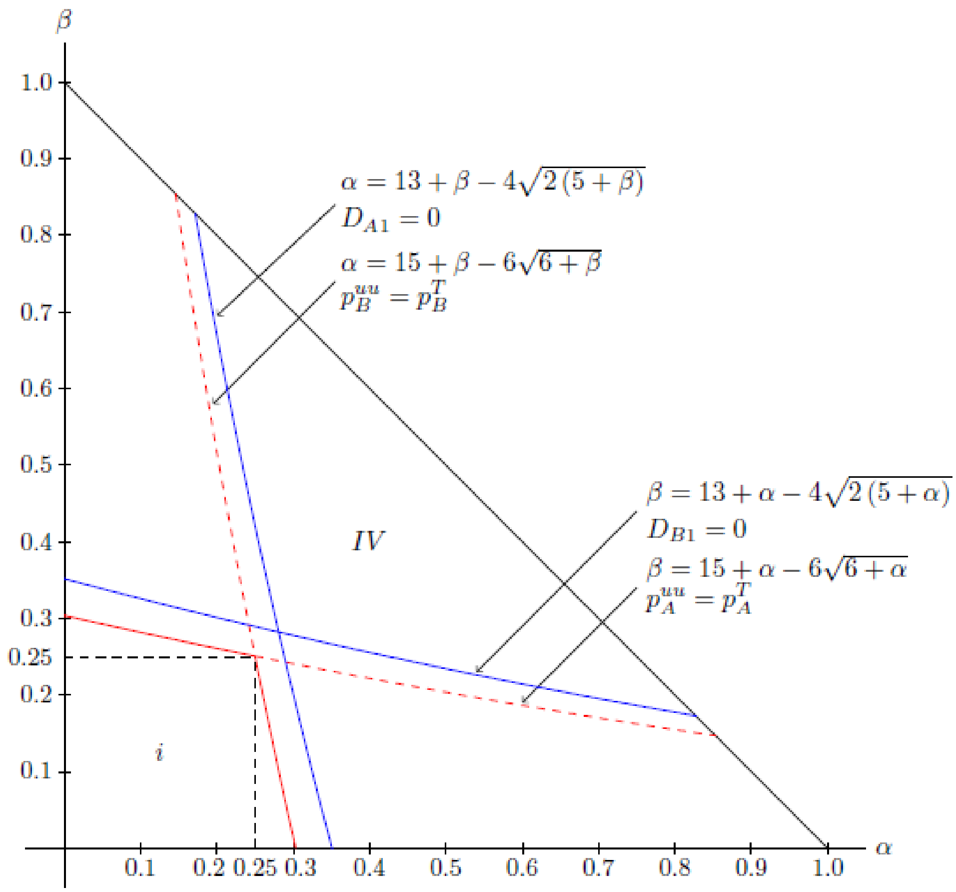

Figure 5: Regions i and IV.

so that firm B is unconstrained, and

$$p_B^{uu} \leq p_B^T, \tag{38}$$

so that firm A is unconstrained. These inequalities define region i. See Figure 5.[13]

In region ii, firm A is unconstrained in equilibrium, firm B is constrained.

In region II, the price best-response function of firm A has two segments; the price best-response function of firm B is always constrained. There are

[13] Shy (2024, Figure 2), Martin (2002, Figure 4.2), and Palander (1935, p. 246, Figure 46). Inequality (37) is equivalent to Shy's (5a).

two possible types of outcomes, *uc* and *cc*. The condition for firm A to be on the unconstrained segment of its price best-response function on region *II* is that B's equilibrium price be weakly less than the threshold value p_B^T, $p_B^{uc} \leq p_B^T$, or

$$\alpha \leq \frac{-1 + 5\sqrt{\beta} - 5\beta + \beta^{\frac{3}{2}}}{1 + \sqrt{\beta}} \tag{39}$$

in terms of the underlying parameters of the model.

Three types of equilibria may occur in the part of the location triangle that is in region *I* but not in region *i*: *uc*, *cc*, and *cu*. The condition for firm A's equilibrium best-response price to be unconstrained is again $p_B^{uc} \leq p_B^T$. B's price best-response function has two segments in the region that is in area *I* but not in area *i*. The condition for B's equilibrium price best-response to be constrained in equilibrium in this region, if A is not constrained, is $p_A^{uc} > p_A^T$. In terms of the underlying parameters of the model, $p_A^{uc} > p_A^T$ is equivalent to the opposite of (37), $p_A^{uu} > p_A^T$, and is satisfied in the part of region *I* that is not in *i* and satisfies (39). The boundary line between region *ii* and region *IV* extends through the space that is in region *I* but not in region *i*, ending at the intersection of the lines that are the upper left and lower right boundaries of region *i*.

In region *iii*, firm A is constrained in equilibrium, firm B is not.

In region *III*, the price best-response function of firm A is always constrained, the price best-response function of firm B has two segments. The condition for firm B to be on the unconstrained portion of its price best-response function on region *III* is $p_A^{cu} \leq p_A^T$, or

$$\beta \leq \frac{-1 + 5\sqrt{\alpha} - 5\alpha + \alpha^{\frac{3}{2}}}{1 + \sqrt{\alpha}} \tag{40}$$

in terms of the underlying parameters of the model.

In the part of the location triangle that is in *I* but not in *i* the condition for B's equilibrium best-response to be unconstrained is again $p_A^{cu} \leq p_A^T$. The condition for firm A to be on the constrained portion of its price best-response function is $p_B^{cu} > p_B^T$. In terms of the underlying parameters of the model, $p_B^{cu} > p_B^T$ is equivalent to the opposite of (38), $p_B^{uu} > p_B^T$, and is satisfied in the part of region *I* that is not in *i* and satisfies (40). The boundary line between region *iii* and region *IV* extends through the space that is in region *I* but not in region *i*, ending at the intersection of the lines that are the upper left and lower right boundaries of region *i*.

In the remaining part of the location triangle, region *iv*, both firms' price best-response functions are constrained in equilibrium.

Lemma 2 (Types of equilibrium prices by location) *Second-stage price equilibria are of four types, depending on firms' positions in the location-space triangle* $0 \leq \alpha \leq 1$*,* $0 \leq \beta \leq 1$*,* $\alpha + \beta \leq 1$ *(Figure 6):*

- Region *i*, $p_A^{uu} \leq p_A^T$, $p_B^{uu} \leq p_B^T$: (28) both firms' equilibrium prices unconstrained.
- Region *ii*: firm A's equilibrium price unconstrained, firm B's equilibrium price constrained.
- Region *iii*: firm A's equilibrium price constrained, firm B's equilibrium price unconstrained.
- Region *iv*: both firms' equilibrium prices constrained..

6.1 A sampling of locations

Figure 7 shows eight positions in location space, for which we illustrate the pairs of price best-response functions. Those for Hotelling's first and second examples, H_1 (region *i*) and H_2 (region *iv*) in Figure 7, are shown in Figures 3 and 4, respectively.

6.1.1 Hotelling's examples: mix and match

Figure 8 shows best response functions and equilibrium prices if firm A is at the location Hotelling's first example, $\alpha = \frac{4}{35}$, and firm B is at the location of Hotelling's second example, $\beta = \frac{8}{20}$. These locations place the firms in region *ii.* In second-stage equilibrium firm A is on the unconstrained segment of its best-response function. Firm B's best-response price is subject to the no-undercutting constraint for all p_A.

Similarly, Figure 9 shows best response functions and equilibrium prices if firm A is at the location of Hotelling's second example, $\alpha = \frac{11}{20}$, and firm B is at the location of Hotelling's first example, $\beta = \frac{1}{35}$. These locations place the firms in region *iii.* In second-stage equilibrium it is firm B that is on the unconstrained segment of its best-response function, while firm A's best-response price constrained throughout.

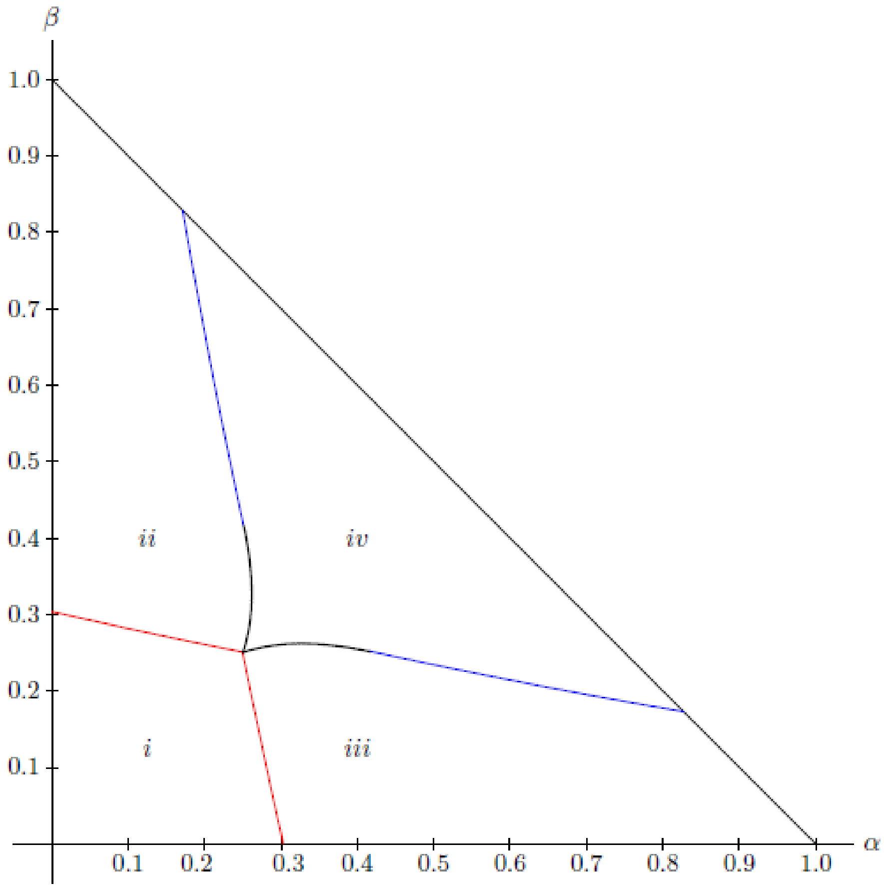

Figure 6: Regions in location space by type of second-stage equilibrium.

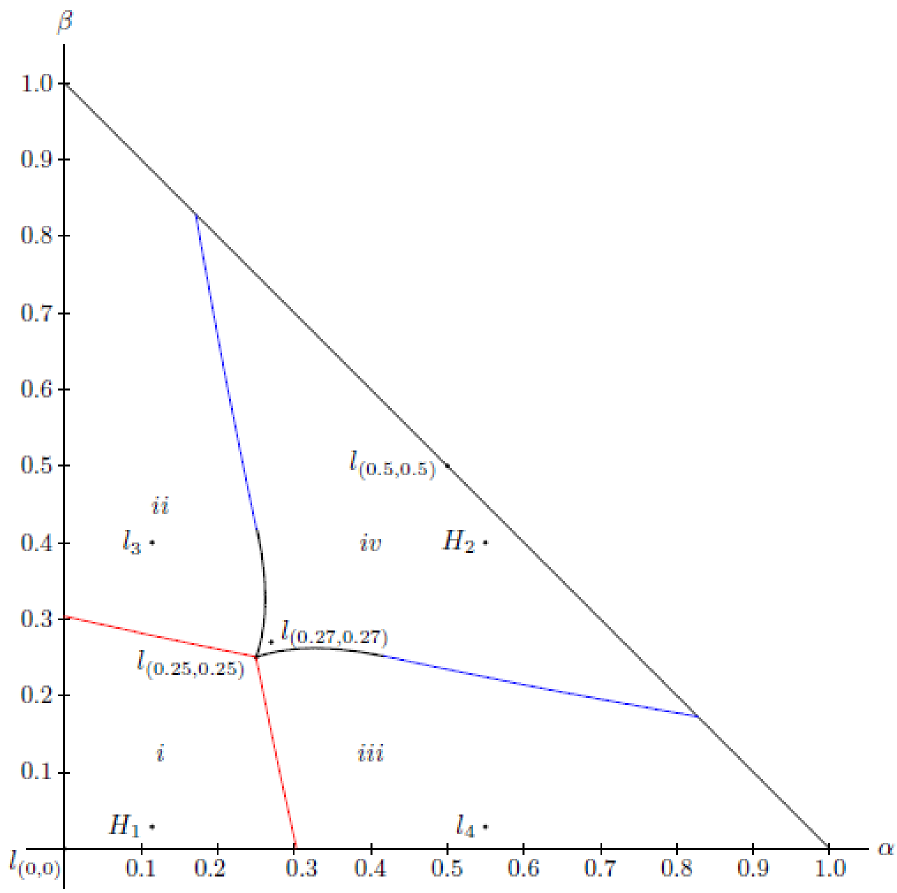

Figure 7: A sample of positions in location space.

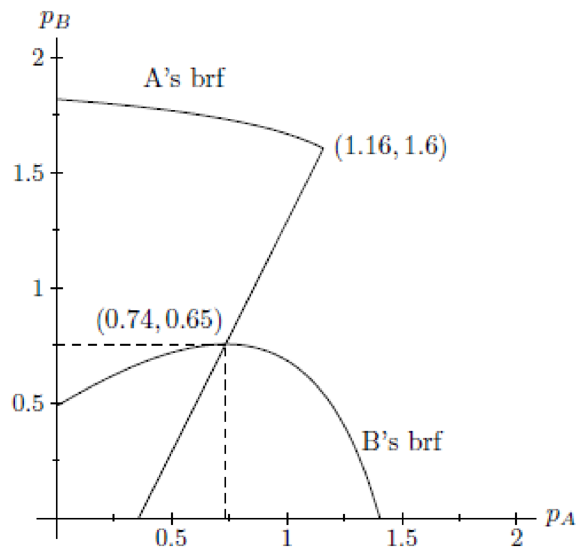

Figure 8: Region *ii* price equilibrium, location l_3 in Figure 7.

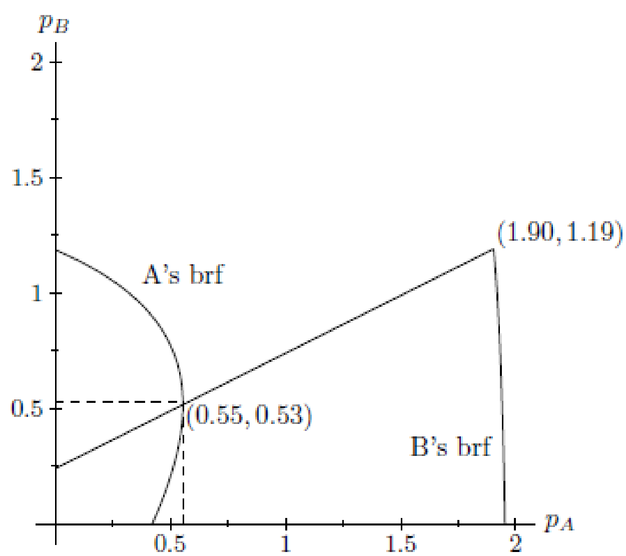

Figure 9: Region *iii* price equilibrium, location l_4 in Figure 7.

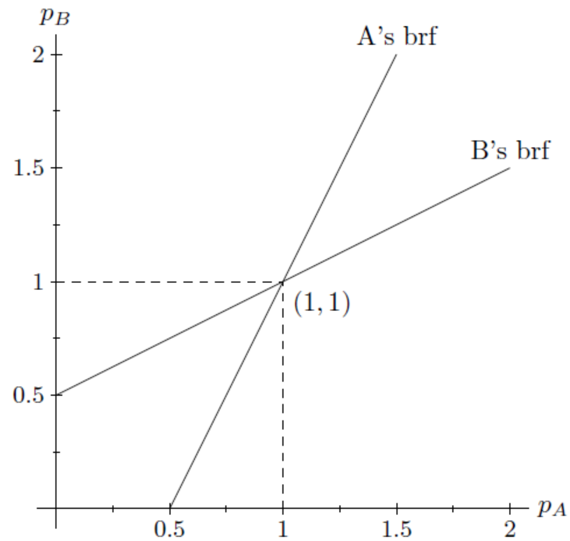

Figure 10: Best-response functions and equilibrium prices, endpoint locations.

6.1.2 Along the diagonal

The origin We next examine equilibria at four locations along the 45-degree line from the origin in location space. The origin itself, $l_{(0,0)}$ in Figure 7, places each firm at one endpoint of the line. See Figure 10. Equilibrium (normalized) prices are 1 (as they are for all symmetric locations in region i; see (28)).

Quartiles At location $l_{(0.25,0.25)}$ in Figure 7, firm are at the quartiles of Main Street. Hotelling singles out the quartile locations as being socially efficient in the sense of minimizing transportation cost. Hamilton *et al.* (1991) and Iskakov and Iskakov (2012) identify quartile locations as equilibrium locations in their generalizations of the Main Street game, as do we in Theorem 3. Undercut-proof best response functions for quartile locations are shown in Figure 11. The best-response functions intersect at the transition prices, again 1, between prices where the no-undercutting constraint does not and does bind.

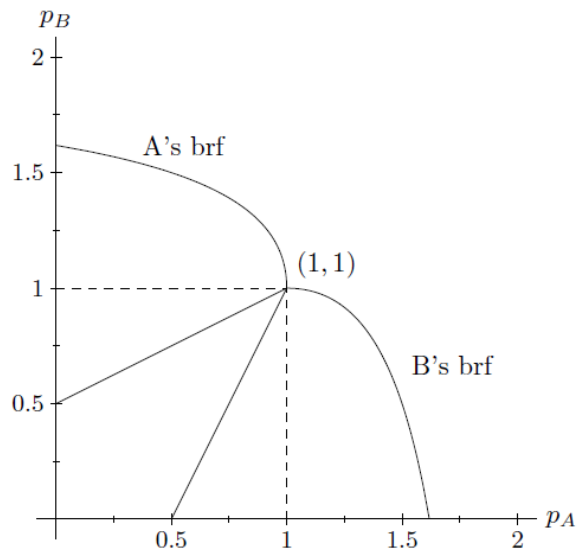

Figure 11: Best-response functions and equilibrium prices, quartile locations.

Just inside the quartiles Figure 12 shows best-response functions and equilibrium prices if firms are located just inside the quartile points. These locations are identified by Osborne and Pitchik (1987), using numerical analysis, as equilibrium locations in the version of the Hotelling game that allows for mixed-strategy pricing. In the present context, this location pair is in region I of Figure 2, so that the best-response functions of both firms have unconstrained and constrained parts, and it is not in region i of Figure 5. With symmetric locations, it is the constrained portions of the best-response functions that intersect and determine equilibrium prices.

The midpoint Figure 13 shows no-undercutting best-response lines and equilibrium prices if firms are located at the center of the line. As expected from the result of d'Aspremont *et al.* (1979), equilibrium prices are 0.

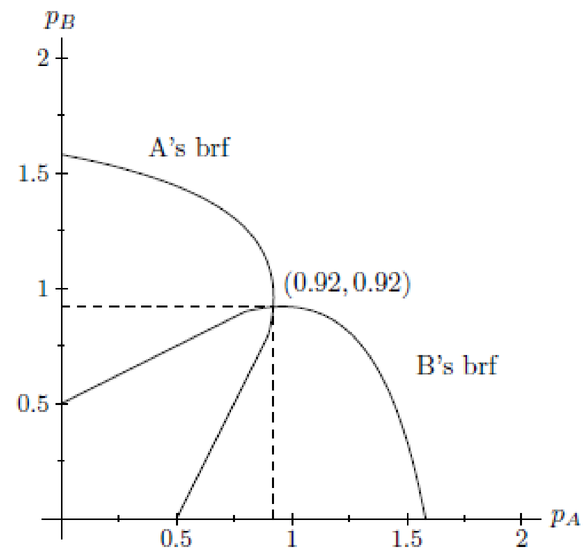

Figure 12: Best-response functions and equilibrium prices, $\alpha = \beta = 0.27$.

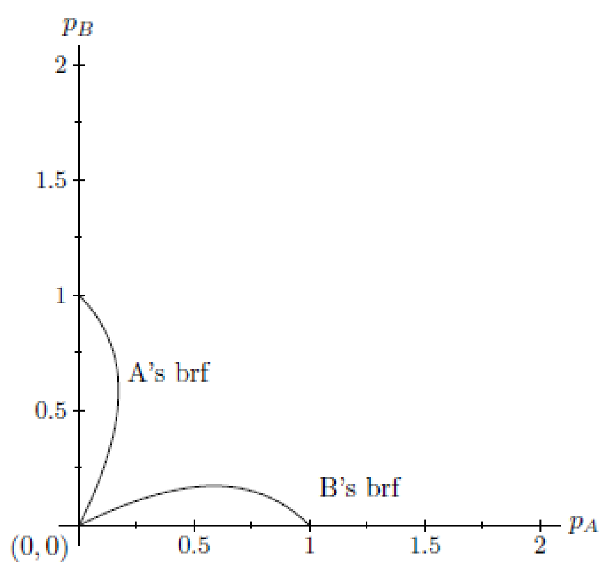

Figure 13: Best-response functions and equilibrium prices, firms located at the midpoint of the line.

7 Location best responses and equilibrium locations

Having worked out equilibrium prices (or, for region *iv*, the conditions that determine equilibrium prices) and payoffs, we turn to the question of noncooperative choice of location in the first stage.

Region *i* payoffs are given by (29). By the argument of Hotelling (1929), π_A^{uu} rises with α. Thus for $0 \leq \beta \leq 0.25$, the local maximum of π_A^{uu} on region *i* is on the region *i*-region *iii* border, and for $0.25 < \beta \leq 0.30306$, the local maximum of π_A^{uu} on region *i* is on the region *i*-region *ii* border.

Region *ii* payoffs are given by (31), from which A's region *ii* payoff is independent of its location in region *ii*. The region *ii*-region *iv* boundary is thus a weak local maximum of π_A^{uc} on region *ii*.

Region *iii* payoffs are given by (33), from which

$$\frac{\partial \pi_A^{cu}}{\partial \alpha} = \frac{1}{2\sqrt{\alpha}} \left(3 + 3\alpha - 8\sqrt{\alpha} - \beta\right). \tag{41}$$

This derivative is negative for $\beta = 0$ and $0.203\,78 \leq \alpha \leq 1$, an interval that contains the range of α on region *iii*, and falls (becomes more negative) as β rises, so is negative in region *iii*. This establishes that for $0 \leq \beta \leq 0.25$ the local maximum of π_A^{cu} on region *iii* is at the region *i*-region *iii* boundary. For $0.25 < \beta \leq 0.261\,60$, the local maximum of π_A^{cu} on region *iii* is on the portion of the region *iii*-region *iv* boundary that is to the left of $(0.327\,18, 0.261\,60)$, the maximum point of the region *iii*-region *iv* boundary.

D'Aspremont *et al.* (1979) observe that $\pi_A^{cc} = \pi_B^{cc} = 0$ on the $\alpha + \beta = 1$ line, the right boundary of region *iv*. From this result it is intuitive that for fixed β π_A^{cc} will initially rise as α falls. Numerical evaluation[14] shows that $\frac{\partial \pi_A^{cc}}{\partial \alpha} < 0$ throughout region *iv*.

Corresponding arguments apply for firm B's location best responses. Then

Theorem 3 *In the first stage of the game, A's location best responses are given by the region i-region iii and region ii-region iv borders in Figure 6 and B's location best responses are given by the region i-region ii and region iii-region iv borders in Figure 6. It follows that noncooperative equilibrium locations are at the quartiles of the normalized Hotelling line,* $\alpha = \beta = \frac{1}{4}$.

[14] The outputs of Maple programs to this effect are available on request from the author.

8 Conclusion

Undercut-proof pricing is a plausible form of behavior to impute to firms. Indeed, in a static two-stage model, it can be thought of as a kind of limit pricing, by each firm against its rival. By formalizing undercut-proof pricing as a constrained optimization problem, we get second-stage price best-response functions that coincide with standard results if the no-undercutting constraints are not binding, and have firms set lower prices if the no-undercutting constraint is binding. We also get first-stage location best-response functions, and equilibrium locations that minimize transportation cost.

9 Appendix A: Normalization

Hotelling's specification is a line of length l, a hinterland of length a for firm A, b for firm B, and constant transportation cost c per unit distance. Normalization of the length of the line to unit length is without loss of generality.

The two equations that determine outcomes if firms share the center region of the line are (i) that delivered price be the same from either firm at the boundary between the markets served by the two firms and (ii) the length of line identity. Divide through the boundary equation by cl to obtain

$$\frac{p_A}{cl} + \frac{x}{l} = \frac{p_B}{cl} + \frac{y}{l}.$$

Divide through the length-of-line identity by l and write $\alpha = \frac{a}{l}$, $\beta = \frac{b}{l}$ to obtain

$$\alpha + \frac{x}{l} + \frac{y}{l} + \beta = 1.$$

So normalized, the two equations solve for normalized sales in the middle of the market,

$$\frac{x}{l} = \frac{1}{2}\left(1 - \alpha - \beta + \frac{p_B}{cl} - \frac{p_A}{cl}\right)$$
$$\frac{y}{l} = \frac{1}{2}\left(1 - \alpha - \beta + \frac{p_A}{cl} - \frac{p_B}{cl}\right).$$

For $\widehat{p}_A = \frac{p_A}{cl}$, $\widehat{p}_B = \frac{p_B}{cl}$, normalized quantities demanded are

$$\widehat{q}_A = \frac{a+x}{l} = \frac{1}{2}\left(1 + \alpha - \beta + \widehat{p}_B - \widehat{p}_A\right)$$

$$\widehat{q}_B = \frac{q_B}{l} = \frac{1}{2}\left(1 - \alpha + \beta + \widehat{p}_A - \widehat{p}_B\right)$$

and normalized objective functions are

$$\widehat{\pi}_A = \frac{\pi_A}{cl^2} = \widehat{p}_A \widehat{q}_A$$

$$\widehat{\pi}_B = \frac{\pi_B}{cl^2} = \widehat{p}_B \widehat{q}_B.$$

Solution values for arbitrary a, b, c and l can be recovered from results for the corresponding normalized specification.

10 Appendix B: A's constrained optimization problem

The Karush-Kuhn-Tucker conditions for the firm A Lagrangian (7) are (8)-(13).

10.1 No-undercutting constraint not binding

Undercutting is not possible for $p_A \leq 1 - \alpha - \beta$. If $p_A > 1 - \alpha - \beta$, it may be that it is possible for B to undercut A's price at A's location, but that undercutting is not B's most profitable option. Suppose the no-undercutting constraint is not binding:

$$\frac{1}{2} p_B \left(1 - \alpha + \beta + p_A - p_B\right) > p_A - \left(1 - \alpha - \beta\right).$$

Then (12) implies $\lambda_A = 0$. Then (9) implies, familiarly,

$$p_A = \frac{1}{2}\left(1 + \alpha - \beta + p_B\right). \tag{42}$$

This is the equation of A's best-response function if the no-undercutting constraint is not binding.

We have assumed $p_A > 1 - \alpha - \beta$. Substitute A's best-response price if the no-undercutting constraint is not binding, (42), in this inequality and rearrange terms to obtain the condition that must be satisfied by p_B for (42) to solve (7).

$$p_B > 2\left(1 - \alpha - \beta\right) - \left(1 + \alpha - \beta\right) = 1 - 3\alpha - \beta. \tag{43}$$

The sharing quantity demanded of B if A sets its best-response price is

$$q_B = \frac{1}{2}(1-\alpha+\beta+p_A-p_B) = \frac{1}{2}\left(1-\alpha+\beta+\frac{1}{2}(1+\alpha-\beta+p_B)-p_B\right)$$
$$= \frac{1}{4}(3-\alpha+\beta-p_B).$$

B's sharing payoff if A sets its best-response price is

$$\frac{1}{4}p_B(3-\alpha+\beta-p_B).$$

Consistency with the KKT conditions for the no-undercutting constraint not to bind A's best response price requires

$$\frac{1}{4}p_B(3-\alpha+\beta-p_B) \geq p_A-(1-\alpha-\beta) \tag{44}$$

for p_A given by (42).

Now find conditions for this inequality to hold. Substitute for A's unconstrained best response price on the right in (44):

$$\frac{1}{4}p_B(3-\alpha+\beta-p_B) \geq \frac{1}{2}(1+\alpha-\beta+p_B)-(1-\alpha-\beta)$$

and collect terms to obtain

$$p_B^2-(1-\alpha+\beta)\,p_B \leq 2(1-3\alpha-\beta).$$

Complete the square on the left and rearrange terms to obtain

$$\left(p_B-\frac{1}{2}(1-\alpha+\beta)\right)^2 \leq 2(1-3\alpha-\beta)+\frac{1}{4}(1-\alpha+\beta)^2 = D_{A1}, \tag{45}$$

where D_{A1} is defined in (18).

The LHS of (45) is positive. If $D_{A1} < 0$, the inequality (44) fails, and the no-undercutting constraint is binding on A's optimization problem.

Consider now, then, the case that $D_{A1} \geq 0$, and return to the contrary case below.

Take the square root of both sides of (45). The no-undercutting constraint is not binding for $p_A > 1-\alpha-\beta$ if

$$-\sqrt{D_{A1}} \leq p_B-\frac{1}{2}(1-\alpha+\beta) \leq \sqrt{D_{A1}}$$

or

$$\frac{1}{2}(1-\alpha+\beta)-\sqrt{D_{A1}} \leq p_B \leq \frac{1}{2}(1-\alpha+\beta)+\sqrt{2(1-3\alpha-\beta)+\frac{1}{4}(1-\alpha+\beta)^2} \equiv p_B^T,$$

where p_B^T is given by (14).

The expression on the left is negative, so the left-hand inequality is always met. For $p_A > 1-\alpha-\beta$, the range of p_B for which the no-undercutting constraint is not binding is

$$\max(0, 1-3\alpha-\beta) \leq p_B \leq \max\left(0, p_B^T\right). \tag{46}$$

p_B^T is the threshold value of p_B above which the no-undercutting constraint binds on firm A.

10.2 No-undercutting constraint binding

(46) is derived under the assumption that $D_{A1} \geq 0$. Consider now the case that $D_{A1} < 0$, which leads to

$$-4\sqrt{2(5+\beta)} < \alpha-(\beta+13) < 4\sqrt{2(5+\beta)}$$

or

$$(\beta+13)-4\sqrt{2(5+\beta)} < \alpha < (\beta+13)+4\sqrt{2(5+\beta)}$$

The right-hand side inequality is always satisfied (recall $0 \leq \beta \leq 1-\alpha$), so it is the left-hand side inequality that characterizes location pairs for which the no-undercutting constraint is binding on A. Intuitively, the larger is α, the more sales in A's hinterland B can make by undercutting A's price at A's location, and the more A will have to lower its price to make such undercutting unprofitable, all else equal.

If the no-undercutting constraint is binding, (9) implies

$$\lambda_A = \frac{1+\alpha-\beta+p_B-2p_A}{2-p_B}, \tag{47}$$

and this must be positive for the no-undercutting constraint to bind. One may show that the numerator on the right in (47) is $\left(p_B - \frac{1}{2}(1-\alpha+\beta)\right)^2 - D_{A1}$, so $\lambda_A > 0$ if $p_B > p_B^T$.

$\lambda_A > 0$ and (12) imply that A's best-response price is given by (22).

The value of p_B that makes A's best-response price equal to 0 is p_B^U as given in (15).

Then there are two cases. If $p_B^T \geq 0$, A's best-response function has two segments, with equation (20). If $p_B^T < 0$, the no-undercutting constraint binds A's best response for all relevant p_B, and has equation (22).

The same arguments, *mutatis mutandis*, give the results for firm B.

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